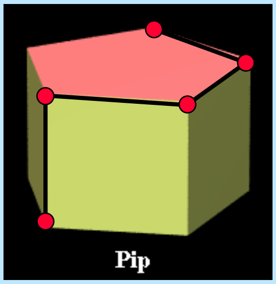
15.053 February 15, 2007

- The Geometry of Linear Programs
  - The simplex algorithm
  - More properties of linear programs



Pentagonal prism

Overview of Lecture

- Review of Geometry
- The Simplex Algorithm
- More on convexity
- RHS Sensitivity Analysis

Quotes of the Day

Geometry is not true, it is advantageous.

Jules H. Poincare

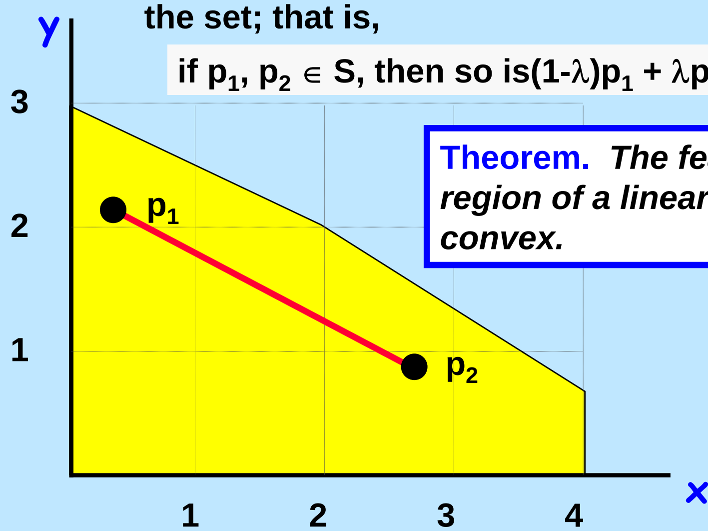
I've always been passionate about geometry and the study of three-dimensional forms.

Erno Rubik

Review of Geometry

A set S is *convex* if for every two points in the set, the line segment joining the points is also in the set; that is,

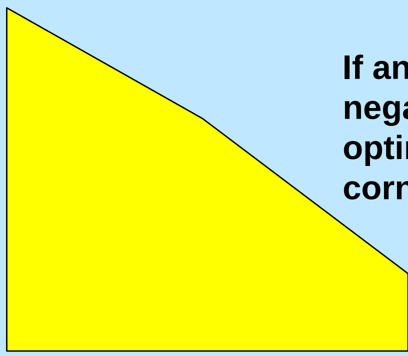
if $p_1, p_2 \in S$, then so is $(1-\lambda)p_1 + \lambda p_2$ for $\lambda \in [0,1]$



Theorem. ***The feasible region of a linear program is convex.***

Corner Points

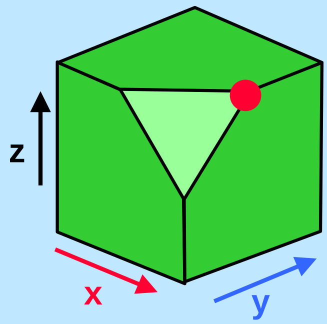
- **A *corner point* of the feasible region is a point that is not the midpoint of two other points of the feasible region.**
- **All feasible LPs with non-negativity constraints have at least one corner point.**



If an LP is feasible, has non-negativity constraints, and has an optimal solution, then there is a corner point that is optimal.

Solving for Corner Points

- In two dimensions, a corner point is the intersection of two equality constraints.
- In three dimensions, a corner point is the intersection of three constraints. (3 planes)



$0 \le x \le 2$

$0 \le y \le 2$

$0 \le z \le 2$

$x - y + z \le 3$

The red corner point is the intersection of three planes

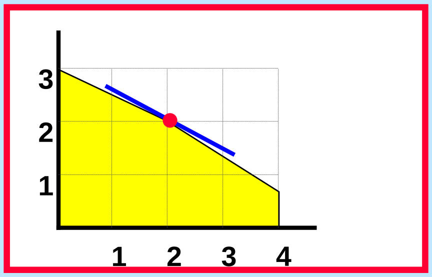
$x = 2$

$z = 2$

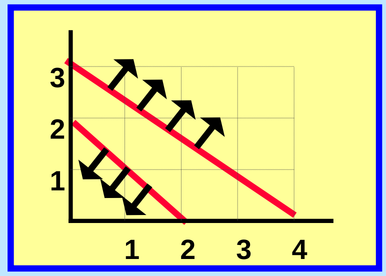
$x - y + z = 3$

The unique solution is $x = 2$, $y = 1$, $z = 2$.

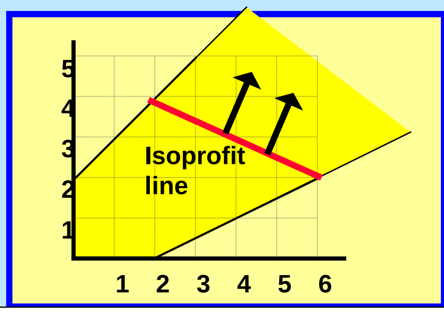
There are 3 Types of Linear Programs



Those with an optimal solution



Those with no feasible solution.



Those whose objective
value is unbounded

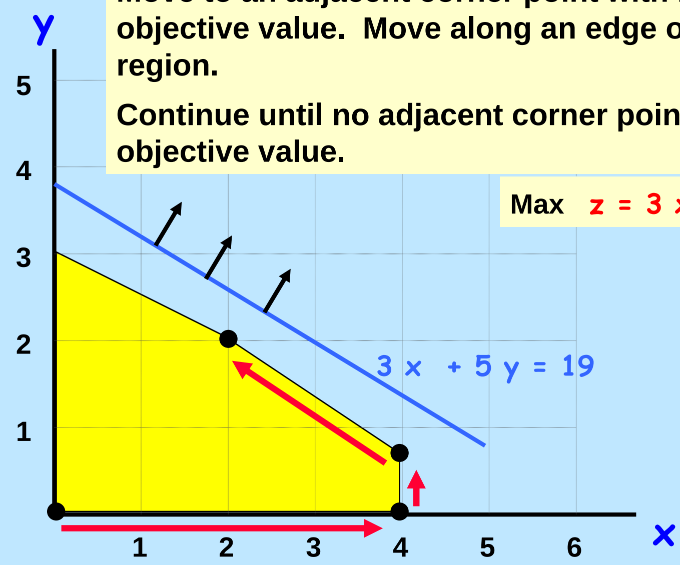
The Simplex Method in Two Dimensions

Start at any feasible corner point.

Move to an adjacent corner point with better objective value. Move along an edge of the feasible region.

Continue until no adjacent corner point has a better objective value.

Max $z = 3x + 5y$

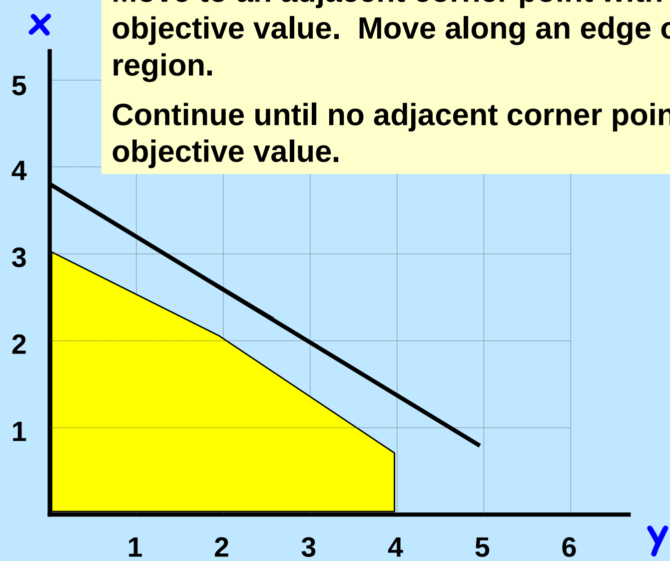


The Simplex Method Again

Start at any feasible corner point.

Move to an adjacent corner point with better objective value. Move along an edge of the feasible region.

Continue until no adjacent corner point has a better objective value.

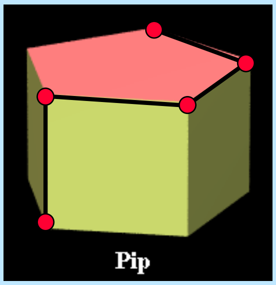


The Simplex Method in 3 Dimensions

Start at any feasible corner point.

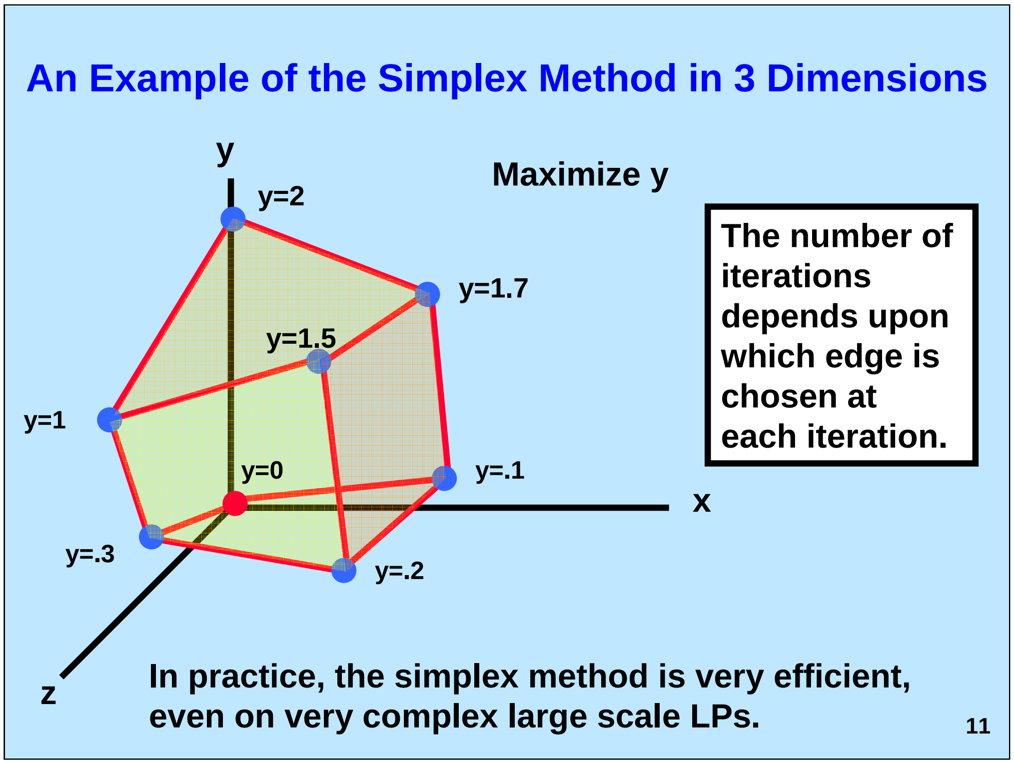
Move to an adjacent corner point with better objective value. Move along an edge of the feasible region.

Continue until no adjacent corner point has a better objective value.



Pentagonal prism

Note: in two dimensions, the "edges" are the intersections of two constraints. The corner points are the intersection of three constraints.



This is a twisted cube.

Notice how the simplex method starting at the origin could move to the optimum in 1 step or pivot.

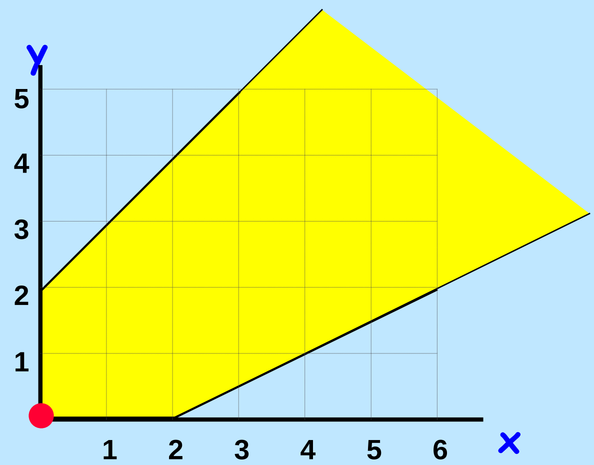
It is also possible for the simplex method to take 7 pivots, thus visiting each corner point.

Klee and Minty developed an example that is very similar that has n variables. It is possible that the simplex method would take 2^n – 1 pivots on these examples, thus showing that the simplex method can take exponential time in the worst case.

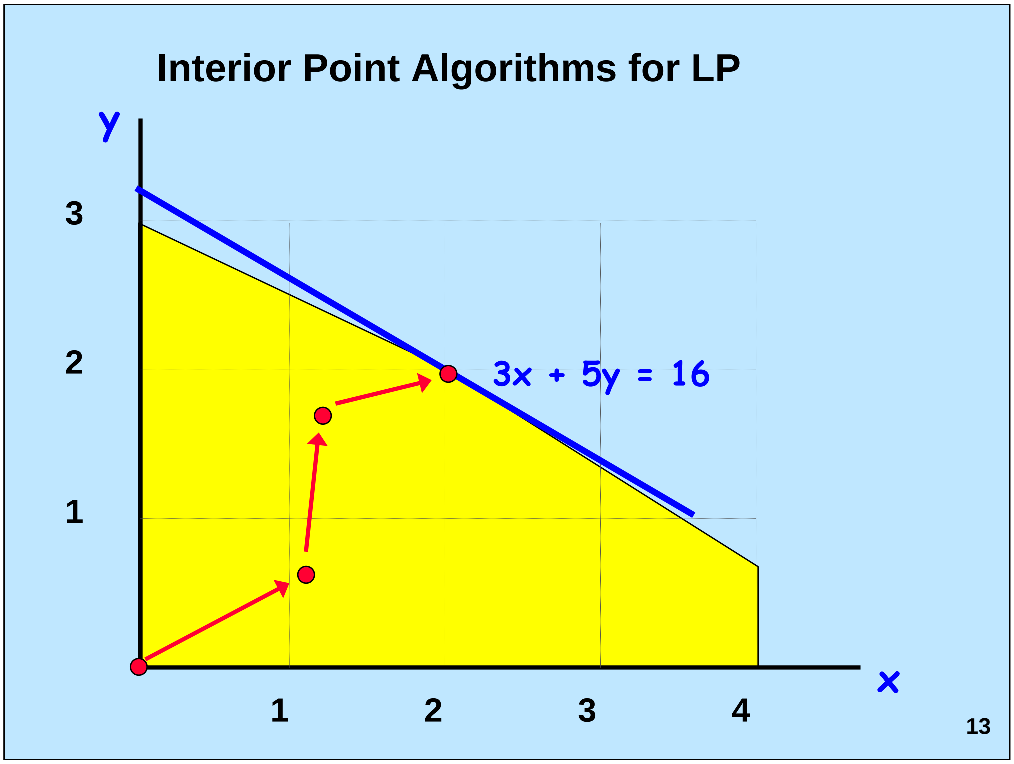
In practice, there may be many different edges that the simplex method can select at a given iteration. The speed in which the simplex method moves to the optimum depends on the choice of the edge.

The Simplex Method on Unbounded LPs

Maximize x



If the objective is unbounded from above, then the simplex method will move infinitely far along an edge.



There are a variety of algorithms that move within the interior of linear programs. These algorithms typically take far fewer iterations than the simplex algorithm and far more time per iteration for large problems. Sometimes interior point algorithms obtain answers quicker than the simplex algorithm. Often they are slower.

Interior point algorithms were popularized by Karmarkar in 1984, who proved that the number of iterations is bounded by a polynomial in the dimension of the problem and in the number of bits needed to describe the coefficients.

While interior point algorithms are ingenious and have practical import, they are also beyond the scope of 15.053, and will not be covered further.

Comments on Optimality Conditions

- Linear programming produces both the optimal solution and the proof of optimality. (This is true for any number of variables, and even if many of the constraints are equality constraints.)
  - special among optimization problems
  - very valuable
  - “The Gold Standard” for optimization

- For other optimization problems in the subject, we will settle for bounds from optimality
  - e.g., we will be happy if we can guarantee at most 10% from optimality

Convex Combinations

Suppose that p_1, p_2, …, p_k are all vectors (or points).

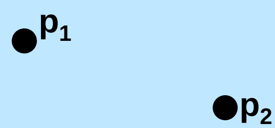
Let $p_{k+1} = \lambda_1 p_1 + \lambda_2 p_2 + ... + \lambda_k p_k$.

We say that p_{k+1} is a convex combination of p_1, …, p_k if the following are true:

$$\lambda_1 + \lambda_2 + ... + \lambda_k = 1$$

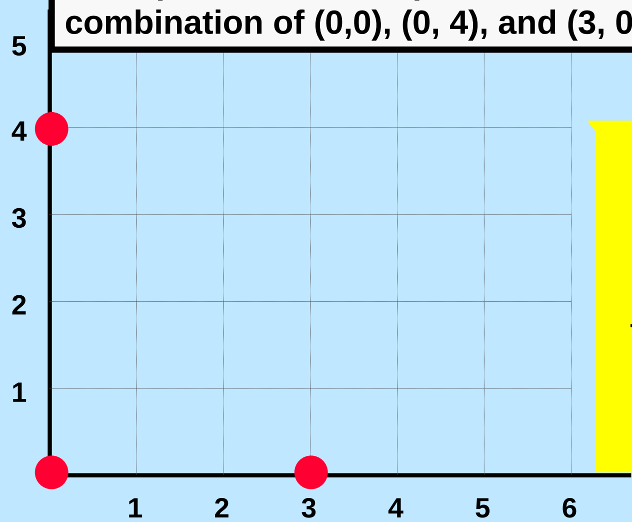
and $\lambda_i \geq 0$ for $i = 1$ to k.

Suppose k = 2. What points are convex combinations of p_1 and p_2?



More on Convex Combinations

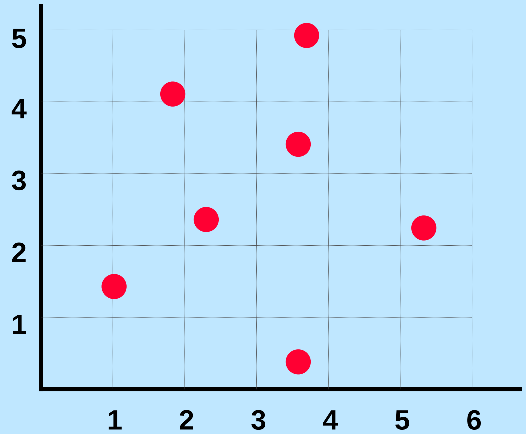
What points can be represented as the convex combination of (0,0), (0, 4), and (3, 0)?



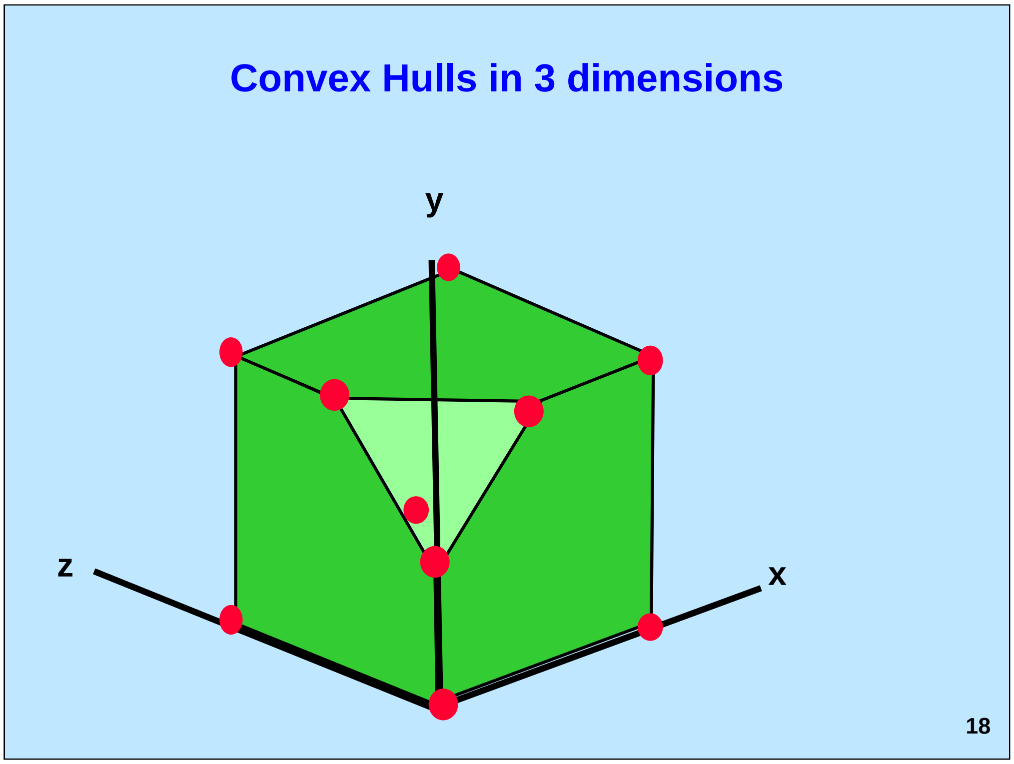
$$\lambda_1 (0, 0) + \lambda_2 (0, 4) + \underline{\lambda_3 (3, 0)}$$

Convex Combinations and Convex Hulls

The convex hull of points p_1, …, p_k is the smallest convex region containing all of the points. It is also the set of all points that can be expressed as convex combinations of p_1 to p_k.



Note that the convex hull of points in 2 dimensions looks like an LP feasible region.



In Slide show mode, the figure is revealed as a cube that is partially cut off.

Representation Theorem

- Theorem.   Every bounded polyhedra (linear programming feasible region) can be represented as a convex hull of its corner points.

- Theorem. The convex hull of a set of points is a bounded linear programming feasible region.

- Usually, we prefer to represent a linear program in terms of constraints. But there are times when it is useful to represent it as the convex combination of corner points.

Mental Break

What are the odds?

Sensitivity Analysis

- **Sensitivity analysis: Determining the marginal effect on the optimal objective function if we make small changes in the data.**

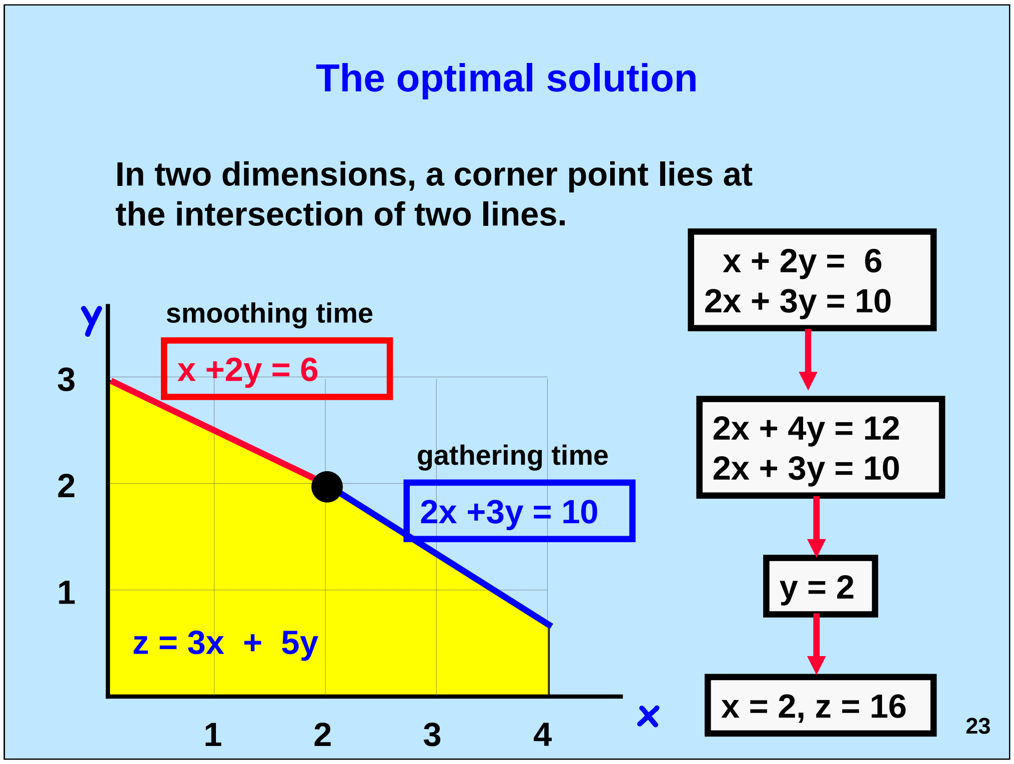
- **In LP, we focus on two types of sensitivity analysis that are very useful and very easy for an LP package to compute**

The revised DTC example

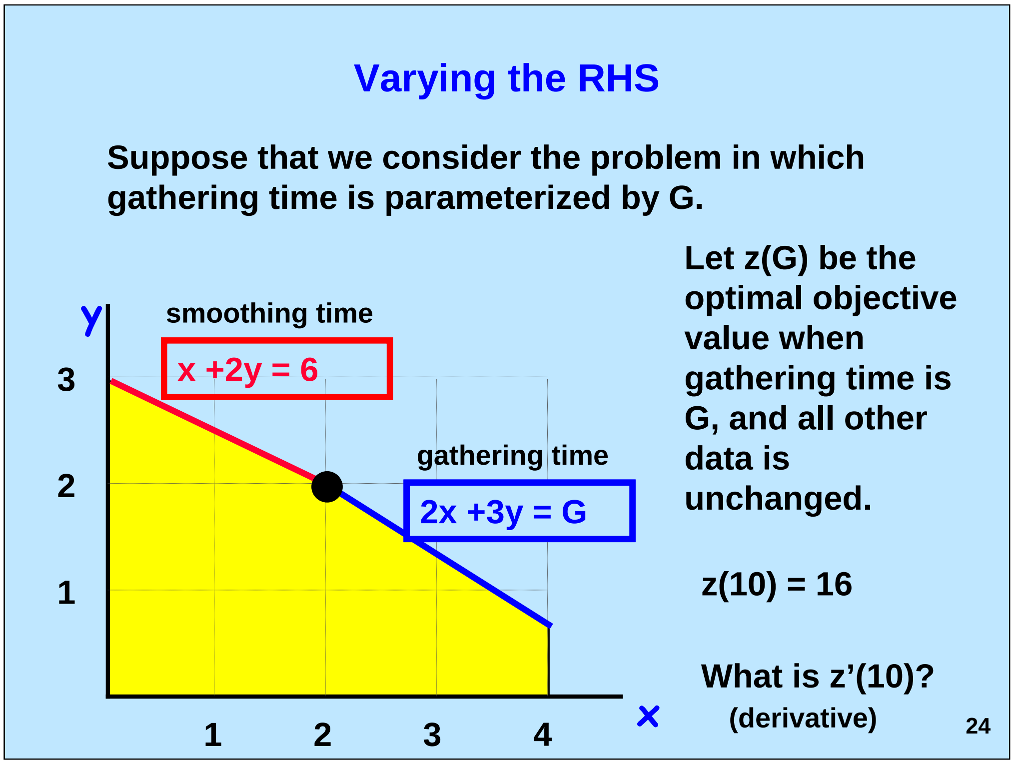
$$z = 3x + 5y$$

Constraint	
$2x + 3y \le 10$	Gathering time
$x + 2y \le 6$	Smoothing time
$x + y \le 5$	Delivery Time
$x \le 4$	Demand: kits
$y \le 3$	Demand: shields
$x, y \ge 0$	non-negativity

We could have used the original variable names of K and S, but it is simpler to use x and y since we usually think of the two axes as the x and y axis.



It's very useful that the corner point lies at the intersection of two lines. Then solving a system of equations with two variables and two equations will give the value of the corner point.



It's very useful that the corner point lies at the intersection of two lines. Then solving a system of equations with two variables and two equations will give the value of the corner point.

Computing the derivative

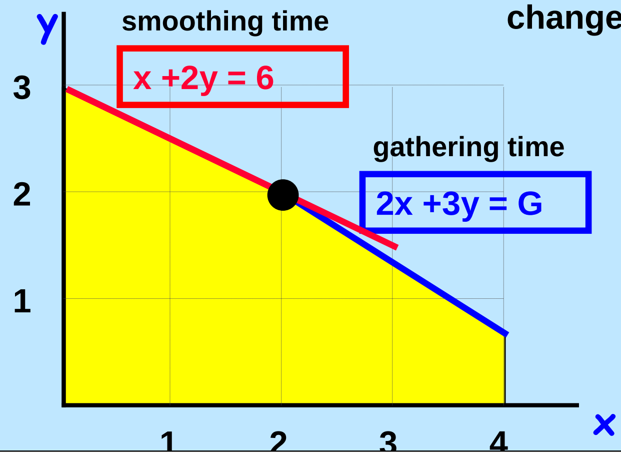
$$z'(10) = \lim_{\Delta \to 0} \frac{z(10+\Delta) - z(10)}{\Delta}$$

Key observation: if Δ is small, then the optimum corner point of the problem will be the intersection of the smoothing time constraint and the gathering constraint.

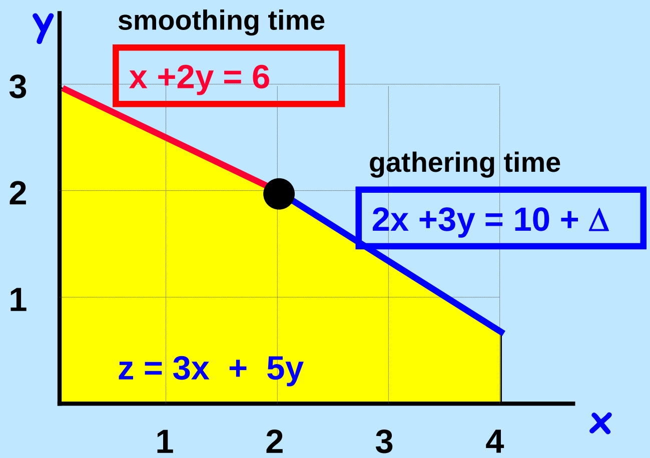
That is, the constraints that define the corner point will not change.

On the new corner point

The solution value changes, but the "structure" of the solution does not change.



Computing the New Corner Point



$$x + 2y = 6$$
$$2x + 3y = 10 + \Delta$$

↓

$$2x + 4y = 12$$
$$2x + 3y = 10 + \Delta$$

↓

$$y = 2 - \Delta$$

↓

$$x = 2 + 2\Delta$$
$$z = 16 + \Delta$$

The Shadow Price

$$z'(10) = \lim_{\Delta \to 0} \frac{z(10+\Delta) - z(10)}{\Delta}$$

$$z'(10) = \lim_{\Delta \to 0} \frac{(16+\Delta) - 16}{\Delta} = 1$$

Note that z'(10 + Δ) is linear in Δ.

This is the *shadow price* of the gathering constraint.

Note. We only needed that the corner point was the intersection of the gathering and smoothing constraint.

More on Shadow Prices

The *shadow price* of a constraint is the unit increase in the optimal objective value per unit increase in the RHS of the constraint. It is also a derivative.

Let p denote the shadow price.

If the RHS of gathering increases from 10 to 10 + Δ, then the objective value increases from 16 to 16 + pΔ, that is, it increases by pΔ.

Exercises

- z(10) = 16;   z'(10) = 1 (The shadow price is 1)
- Fact: z(11) = 16 + 1 = 17.
- What is z(10.2)?
- What is z(9.7)?
- What is z(0)?   (trick question)

Computing the Shadow Price of a Constraint

- Step 1. Determine the binding constraints that determine the corner point.

$$x + 2y = 6$$
$$2x + 3y = 10$$

- Step 2. Add Δ to the RHS of the constraint whose shadow price we are computing.

$$x + 2y = 6$$
$$2x + 3y = 10 + \Delta$$

- Step 3. Solve the system of equations.

$$x = 2 + 2\Delta$$
$$y = 2 - \Delta$$

- Step 4. Compute the "increase" in z when Δ increases from 0 to 1 (i.e., compute the derivative)

$$z(10 + \Delta) = 16 + \Delta$$
Shadow price = 1.

Bounds on RHS coefficients in Sensitivity Analysis

- Recall that the optimum solution is a corner point, which in 2 dimensions is the solution of 2 equations in 2 variables, and the equations are the binding constraints.

- Compute the largest changes in the RHS coefficient so that all constraints remain satisfied.

What happens if the RHS changes by a lot?

More Sensitivity Analysis: Determining the Interval

$x = 2 + 2\Delta$; $y = 2 - \Delta$

		Constraint after substitution.
Maximize Profit	$z = 3x + 5y$ (in 10s)	$z = 16 + \Delta$
Gathering time:	$2x + 3y \le 10 + \Delta$	$10 + \Delta \le 10 + \Delta$
Smoothing time:	$x + 2y \le 6$	$6 \le 6$
Delivery time:	$x + y \le 5$	$4 + \Delta \le 5$
Slingshot demand:	$x \le 4$	$2 + 2\Delta \le 4$
Shield demand:	$y \le 3$	$2 - \Delta \le 3$
Non-negativity:	$K, S \ge 0$	$2 + 2\Delta \ge 0$ $2 - \Delta \ge 0$

So, $-1 \le \Delta \le 1$

Summary for changes in RHS coefficients

- Determine the binding constraints
- Determine the change in the "corner point solution" as a function of Δ.
- Compute the largest and smallest values of Δ so that the solution stays feasible.
- The shadow price is valid so long as the "corner point solution" remains optimal, which is so long as it is feasible.

And now, it's time for .....

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"Who wants a piece of candy" is not stored on the web.