Module – 3 Lecture Notes – 4

Simplex Method – II

Introduction

In the previous lecture the *simplex method* was discussed with required transformation of objective function and constraints. However, all the constraints were of inequality type with 'less-than-equal-to' $(\leq)$ sign. However, 'greater-than-equal-to' $(\geq)$ and 'equality' $(=)$ constraints are also possible. In such cases, a modified approach is followed, which will be discussed in this lecture. Different types of LPP solutions in the context of Simplex method will also be discussed. Finally, a discussion on minimization vs maximization will be presented.

Simplex Method with 'greater-than-equal-to' ($\geq$) and equality ($=$) constraints

The LP problem, with 'greater-than-equal-to' $(\geq)$ and equality $(=)$ constraints, is transformed to its standard form in the following way.

1. One 'artificial variable' is added to each of the 'greater-than-equal-to' $(\geq)$ and equality $(=)$ constraints to ensure an initial basic feasible solution.
2. Artificial variables are 'penalized' in the objective function by introducing a large negative (positive) coefficient M for maximization (minimization) problem.
3. Cost coefficients, which are supposed to be placed in the Z-row in the initial simplex tableau, are transformed by 'pivotal operation' considering the column of artificial variable as 'pivotal column' and the row of the artificial variable as 'pivotal row'.
4. If there are more than one artificial variable, step 3 is repeated for all the artificial variables one by one.

Let us consider the following LP problem

$$
\begin{aligned}
\text{Maximize} \quad & Z = 3x_1 + 5x_2 \\
\text{subject to} \quad & x_1 + x_2 \geq 2 \\
& x_2 \leq 6 \\
& 3x_1 + 2x_2 = 18 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

After incorporating the artificial variables, the above LP problem becomes as follows:

$$
\begin{array}{ll}
\text{Maximize} & Z = 3x_1 + 5x_2 - Ma_1 - Ma_2 \\
\text{subject to} & x_1 + x_2 - x_3 + a_1 = 2 \\
& x_2 + x_4 = 6 \\
& 3x_1 + 2x_2 + a_2 = 18 \\
& x_1, x_2 \geq 0
\end{array}
$$

where x_3 is surplus variable, x_4 is slack variable and a_1 and a_2 are the artificial variables. Cost coefficients in the objective function are modified considering the first constraint as follows:

$$Z - 3x_1 - 5x_2 + Ma_1 + Ma_2 = 0 \qquad (E_1)$$

$$x_1 + x_2 - x_3 + a_1 = 2 \qquad (E_2) \leftarrow \text{Pivotal Row}$$

Pivotal Column (a_1)

Thus, pivotal operation is $E_1 - M \times E_2$, which modifies the cost coefficients as follows:

$$Z - (3+M)x_1 - (5+M)x_2 + Mx_3 + 0\,a_1 + Ma_2 = -2M$$

Next, the revised objective function is considered with third constraint as follows:

$$Z - (3+M)x_1 - (5+M)\,x_2 + Mx_3 + 0\,a_1 + Ma_2 = -2M \qquad (E_3)$$

$$3\,x_1 + 2\,x_2 + a_2 = 18 \qquad (E_4) \leftarrow \text{Pivotal Row}$$

Pivotal Column (a_2)

Obviously pivotal operation is $E_3 - M \times E_4$, which further modifies the cost coefficients as follows:

$$Z - (3+4M)x_1 - (5+3M)x_2 + Mx_3 + 0\,a_1 + 0\,a_2 = -20M$$

The modified cost coefficients are to be used in the Z-row of the first simplex tableau.

Next, let us move to the construction of simplex tableau. Pivotal column, pivotal row and pivotal element are marked (same as used in the last class) for the ease of understanding.

Iteration	Basis	Z	Variables						b_r	$\frac{b_r}{c_{rs}}$
			x_1	x_2	x_3	x_4	a_1	a_2		
1	Z	1	$-3-4M$	$-5-3M$	M	0	0	0	$-20M$	--
	a_1	0	1	1	-1	0	1	0	2	2
	x_4	0	0	1	0	1	0	0	6	--
	a_2	0	3	2	0	0	0	1	18	6

Note that while comparing $(-3-4M)$ and $(-5-3M)$, it is decided that $(-3-4M)<(-5-3M)$ as M is any arbitrarily large number.

Successive iterations are shown as follows:

Iteration	Basis	Z	Variables						b_r	$\frac{b_r}{c_{rs}}$
			x_1	x_2	x_3	x_4	a_1	a_2		
2	Z	1	0	$-2+M$	$-3-3M$	0	$3+4M$	0	$6-12M$	--
	x_1	0	1	1	-1	0	1	0	2	--
	x_4	0	0	1	0	1	0	0	6	--
	a_2	0	0	-1	3	0	-3	1	12	4

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Iteration	Basis	Z	x_1	x_2	x_3	x_4	a_1	a_2	b_r	$\frac{b_r}{c_{rs}}$
3	Z	1	0	-3	0	0	M	$1+M$	18	--
	x_1	0	1	$\frac{2}{3}$	0	0	0	$\frac{1}{3}$	6	9
	x_4	0	0	1	0	1	0	0	6	6
	x_3	0	0	$-\frac{1}{3}$	1	0	-1	$\frac{1}{3}$	4	--
4	Z	1	0	0	0	3	M	$1+M$	36	--
	x_1	0	1	0	0	$-\frac{2}{3}$	0	$\frac{1}{3}$	2	--
	x_2	0	0	1	0	1	0	0	6	--
	x_3	0	0	0	1	$\frac{1}{3}$	-1	$\frac{1}{3}$	6	--

It is found that, at iteration 4, optimality has reached. Optimal solution is $Z = 36$ with $x_1 = 2$ and $x_2 = 6$. The methodology explained above is known as *Big-M* method. Hope, reader has already understood the meaning of the terminology!

‘Unbounded’, ‘Multiple’ and ‘Infeasible’ solutions in the context of Simplex Method

As already discussed in lecture notes 2, a linear programming problem may have different type of solutions corresponding to different situations. Visual demonstration of these different types of situations was also discussed in the context of graphical method. Here, the same will be discussed in the context of Simplex method.

Unbounded solution

If at any iteration no departing variable can be found corresponding to entering variable, the value of the objective function can be increased indefinitely, i.e., the solution is unbounded.

Multiple (infinite) solutions

If in the final tableau, one of the non-basic variables has a coefficient 0 in the Z-row, it indicates that an alternative solution exists. This non-basic variable can be incorporated in the basis to obtain another optimal solution. Once two such optimal solutions are obtained, infinite number of optimal solutions can be obtained by taking a weighted sum of the two optimal solutions.

Consider the slightly revised above problem,

$$\begin{aligned} \text{Maximize} \quad & Z = 3x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + x_2 \geq 2 \\ & x_2 \leq 6 \\ & 3x_1 + 2x_2 = 18 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Curious readers may find that the only modification is that the coefficient of x_2 is changed from 5 to 2 in the objective function. Thus the slope of the objective function and that of third constraint are now same. It may be recalled from lecture notes 2, that if the *Z line* is parallel to any side of the feasible region (i.e., one of the constraints) all the points lying on that side constitute optimal solutions (refer fig 3 in lecture notes 2). So, reader should be able to imagine graphically that the LPP is having infinite solutions. However, for this particular set of constraints, if the objective function is made parallel (with equal slope) to either the first constraint or the second constraint, it will not lead to multiple solutions. The reason is very simple and left for the reader to find out. As a hint, plot all the constraints and the objective function on an arithmetic paper.

Now, let us see how it can be found in the simplex tableau. Coming back to our problem, final tableau is shown as follows. Full problem is left to the reader as practice.

Final tableau:

Iteration	Basis	Z	Variables						b_r	$\frac{b_r}{c_{rs}}$
			x_1	x_2	x_3	x_4	a_1	a_2		
3	Z	1	0	0	0	0	M	$1+M$	18	--
	x_1	0	1	$\frac{2}{3}$	0	0	0	$\frac{1}{3}$	6	9
	x_4	0	0	1	0	1	0	0	6	6
	x_3	0	0	$-\frac{1}{3}$	1	0	-1	$\frac{1}{3}$	4	--

Coefficient of non-basic variable x_2 is zero

As there is no negative coefficient in the Z-row the optimal is reached. The solution is $Z=18$ with $x_1=6$ and $x_2=0$. However, the coefficient of non-basic variable x_2 is zero as shown in the final simplex tableau. So, another solution is possible by incorporating x_2 in the basis. Based on the $\frac{b_r}{c_{rs}}$, x_4 will be the exiting variable. The next tableau will be as follows:

Iteration	Basis	Z	Variables						b_r	$\frac{b_r}{c_{rs}}$
			x_1	x_2	x_3	x_4	a_1	a_2		
4	Z	1	0	0	0	0	M	$1+M$	18	--
	x_1	0	1	0	0	$-\frac{2}{3}$	0	$\frac{1}{3}$	2	--
	x_2	0	0	1	0	1	0	0	6	6
	x_3	0	0	0	1	$\frac{1}{3}$	-1	$\frac{1}{3}$	6	18

Coefficient of non-basic variable x_4 is zero

Thus, another solution is obtained, which is $Z=18$ with $x_1=2$ and $x_2=6$. Again, it may be noted that, the coefficient of non-basic variable x_4 is zero as shown in the tableau. If one more similar step is performed, same simplex tableau at iteration 3 will be obtained.

Thus, we have two sets of solutions as $\begin{Bmatrix} 6 \\ 0 \end{Bmatrix}$ and $\begin{Bmatrix} 2 \\ 6 \end{Bmatrix}$. Other optimal solutions will be obtained as $\beta \begin{Bmatrix} 6 \\ 0 \end{Bmatrix} + (1-\beta) \begin{Bmatrix} 2 \\ 6 \end{Bmatrix}$ where, $\beta \in [0,1]$. For example, let $\beta = 0.4$, corresponding solution is $\begin{Bmatrix} 3.6 \\ 3.6 \end{Bmatrix}$, i.e., $x_1 = 3.6$ and $x_2 = 3.6$. Note that values of the objective function are not changed for different sets of solution; for all the cases $Z = 18$.

Infeasible solution

If in the final tableau, at least one of the artificial variables still exists in the basis, the solution is indefinite.

Reader may check this situation both graphically and in the context of Simplex method by considering following problem:

$$\begin{aligned} \text{Maximize} \quad & Z = 3x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + x_2 \le 2 \\ & 3x_1 + 2x_2 \ge 18 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Minimization versus maximization problems

As discussed earlier, standard form of LP problems consist of a maximizing objective function. Simplex method is described based on the standard form of LP problems, i.e., objective function is of maximization type. However, if the objective function is of minimization type, simplex method may still be applied with a small modification. The required modification can be done in either of following two ways.

1. The objective function is multiplied by -1 so as to keep the problem identical and 'minimization' problem becomes 'maximization'. This is because of the fact that minimizing a function is equivalent to the maximization of its negative.
2. While selecting the entering nonbasic variable, the variable having the maximum coefficient among all the cost coefficients is to be entered. In such cases, optimal

solution would be determined from the tableau having all the cost coefficients as non-positive (≤ 0)

Still one difficulty remains in the minimization problem. Generally the minimization problems consist of constraints with 'greater-than-equal-to' ($\geq$) sign. For example, minimize the price (to compete in the market); however, the profit should cross a minimum threshold. Whenever the goal is to minimize some objective, lower bounded requirements play the leading role. Constraints with 'greater-than-equal-to' ($\geq$) sign are obvious in practical situations.

To deal with the constraints with 'greater-than-equal-to' ($\geq$) and $=$ sign, *Big-M* method is to be followed as explained earlier.