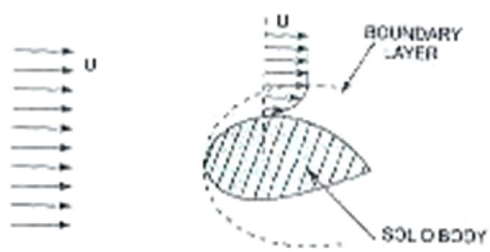
BOUNDARY LAYER CONCEPT IN THE STUDY OF FLUID FLOW

When fluids flow over surfaces, the molecules near the surface are brought to rest due to the viscosity of the fluid. The adjacent layers are also slow down, but to a lower and lower extent. This slowing down is found limited to a thin layer near the surface. The fluid beyond this layer is not affected by the presence of the surface. The fluid layer near the surface in which there is a general slowing down is defined as boundary layer. The velocity of flow in this layer increases from zero at the surface to free stream velocity at the edge of the boundary layer.

When a real fluid flow past a solid body or a solid wall, the fluid particles adhere to the boundary and condition of no slip occurs. This means that the velocity of fluid close to the boundary will be same as that of the boundary. If the boundary is stationary, the velocity of fluid at the boundary will be zero. The theory dealing with boundary layer flows is called boundary layer theory.

According to the B.L. theory, the flow of fluid in the neighbourhood of the solid boundary may be divided into two regions as shown below

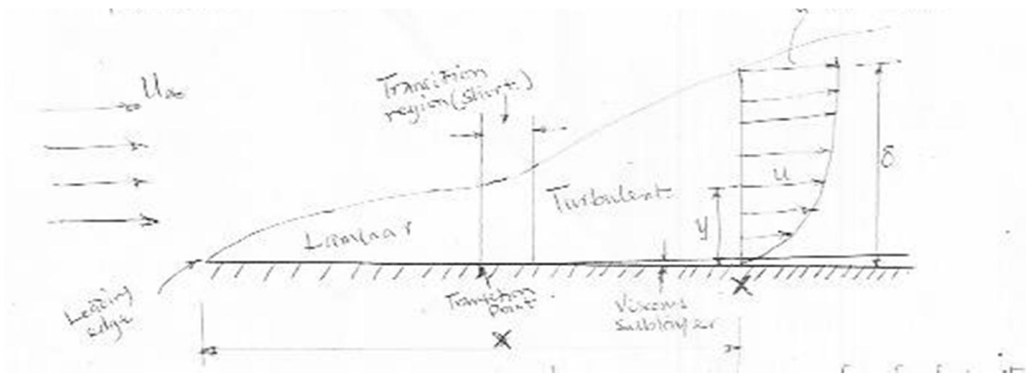


Description of the Boundary Layer

The simplest boundary layer to study is that formed in the flow along one side of a thin, smooth, flat plate parallel to the direction of the oncoming fluid. No other solid surface is near, and the pressure of the fluid is uniform. If the fluid were inviscid no velocity gradient would, in this instance, arise. The velocity gradients in a real fluid are therefore entirely due to viscous action near the surface.

The fluid, originally having velocity U_∞ in the direction of plate, is retarded in the neighbourhood of the surface, and the boundary layer begins at the leading edge of the plate. As more and more of the fluid is slowed down, the thickness of the layer increases. The fluid in contact with the plate surface has zero velocity, 'no slip' and a velocity gradient exists between the fluid in the free stream and the plate surface.

The flow in the first part of the boundary layer (close to the leading edge of the plate) is entirely laminar. With increasing thickness, however, the laminar layer becomes unstable, and the motion within it becomes disturbed. The irregularities of the flow develop into turbulence, and the thickness of the layer increases more rapidly. The changes from laminar to turbulent flow take place over a short length known as the transition region.



Graph of velocity u against distance y from surface at point X

Reynolds' Number Concept

If the Reynolds number locally were based on the distance from the leading edge of the plate, then it will be appreciated that, initially, the value is low, so that the fluid flow close to the wall may be categorized as laminar. However, as the distance from the leading edge increases, so does the Reynolds number until a point is reached where the flow regime becomes turbulent.

For smooth, polished plates the transition may be delayed until Re equals 500000. However, for rough plates or for turbulent approach flows transition may occur at much lower values. Again, the transition does not occur in practice at one well-defined point but, rather, a transition zone is established between the two flow regimes.

The figure above also depicts the distribution of shear stress along the plate in the flow direction. At the leading edge, the velocity gradient is large, resulting in a high shear stress. However, as the laminar region progresses, so the velocity gradient and shear stress decrease with thickening of the boundary layer. Following transition the velocity gradient again increases and the shear stress rises.

Theoretically, for an infinite plate, the boundary layer goes on thickening indefinitely. However, in practice, the growth is curtailed by other surfaces in the vicinity.

Factors affecting transition from Laminar to Turbulent flow Regimes

As mentioned earlier, the transition from laminar to turbulent boundary layer condition may be considered as Reynolds number dependent, $\mathrm{Re}_x = \rho U_s x / \mu = \frac{\mu x}{\nu}$ and a figure of 5×10^5 is often quoted.

However, this figure may be considerably reduced if the surface is rough. For Re $<10^5$, the laminar layer is stable; however, at Re near 2 x 10^5 it is difficult to prevent transition.

The presence of a pressure gradient dp/dx can also be a major factor. Generally, if dp/dx is positive, then transition Reynolds number is reduced, a negative dp/dx increasing transition Reynolds number.

Boundary Layer thickness (σ)

The velocity within the boundary layer increases from zero at the boundary surface to the velocity of the main stream asymptotically. Therefore the thickness of the boundary layer is arbitrarily defined as that distance from the boundary in which the velocity reaches 99 per cent of the velocity of the velocity of the free stream ($u = 0.99U_\infty$). It is denoted by the symbol σ. This definition however gives an approximate value of the boundary layer thickness and hence σ is generally termed as nominal thickness of the boundary layer.

The boundary layer thickness for greater accuracy is defined as in terms of certain mathematical expression which are the measure of the boundary layer on the flow. The commonly adopted definitions of the boundary layer thickness are:

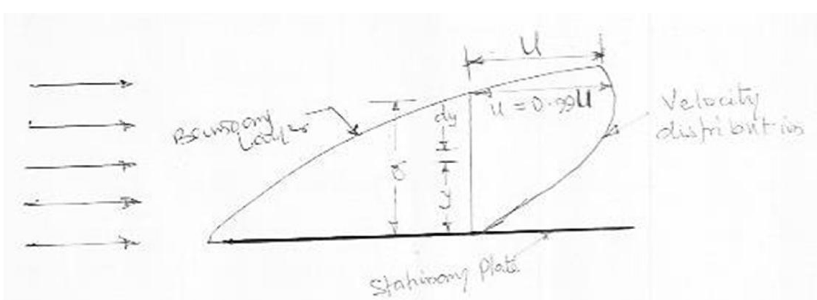
1. Displacement thickens (σ^*)
2. Momentum thickness (θ)
3. Energy thickness (∂_c)

- **Displacement thickness (σ^*)**

The displacement thickness can be defined as the distance measured perpendicular to the boundary by which the main/free stream is displaced on account of formation boundary layer.

Or

It is an additional "Wall thickness" that would have to be added to compensate for the reduction in flow rate on account of boundary layer formation".



Displacement thickness

Let fluid of density ℓ flow past a stationary plate with velocity U as shown above. Consider an elementary strip of thickness dry at a distance y from the plate.

Assumed unit width, the mass flow per second through the elementary strip

$$= \ell udy ---------------- (i)$$

Mass of flow per second through the elementary strip (unit width) if the plate were not there

$$= \ell udy ---------------- (ii)$$

Reduce the mass flow rate through the elementary strip

$$= \ell udy - \ell udy$$
$$= \ell (u - u) dy$$

Total momentum of mass flow rate due to introduction of plate

$$= \int_0^{\delta} \rho (U - u) dy ----------------- (iii)$$

(If the fluid is incompressible)

Let the plate is displaced by a distance σ^* and velocity of flow for the distance σ^* is equal to the main/free stream velocity (i.e. U). Then, loss of the mass of the fluid/sec. flowing through the distance σ^*.

$$= \rho U \sigma^* ------------ (iv)$$

Equating eqns. (iii) and (iv) we get

$$= \rho U \sigma^* = \int_0^{\sigma} \rho (U - u) dy$$

or

$$\sigma^* = \int_0^{\sigma} \left(1 - \frac{u}{U}\right) dy$$

Momentum Thickness (θ)

This is defined as the distance which the total loss of momentum per second be equal to if it were passing a stationary plate. It is denoted by θ.

It may also be defined as the distance, measured perpendicular to the boundary of the solid body by which the boundary should be displaced to compensate for reduction in momentum of the flowing fluid on account of boundary layer formation.

Refer to diagram of displacement thickness above, Mass of flow per second through the elementary strip = $\rho u dy$

Momentum/Sec. of this fluid inside the boundary layer

$= \rho u dy \times U = \rho u^2 dy$

Momentum/sec. of the same mass of fluid before entering boundary layer = $\rho u U dy$

Loss of Momentum/sec. = $\rho u U dy - \rho u^2 dy = \rho u (U - u) dy$

$\therefore$ Total loss of momentum/sec

$$= \int_0^\delta \rho u (U - u) dy \text{ ---------- } (i)$$

Let θ = Distance by which plate is displaced when the fluid is flowing with a constant velocity U. then loss of momentum/Sec. of fluid flowing through distance θ with a velocity U.

$$= \rho \theta U^2 \text{ ------------ } (ii)$$

Equating eqns. (i) and (ii), we have

$$\rho \theta u^2 = \int_0^\delta \rho u (U - u) dy$$

$$OR$$

$$\theta = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

Energy Thickness (∂_e)

Energy thickness is defined as the distance measured perpendicular to the boundary of the solid body, by which the boundary should be displaced to compensate for the reduction in K.E of the flowing fluid on account of boundary layer formation. It is denoted by (∂_e)

Refer to the above displacement thickness diagram,

Mass of flow per second through the elementary strip = $\rho u dy$

K.E of this fluid inside the boundary layer = $\frac{1}{2} m u^2 = \frac{1}{2}(\rho u dy) u^2$

K.E of the same mass of fluid before entering the boundary layer

$$\tfrac{1}{2}(\rho u dy) u^2$$

Loss of K.E. through elementary strip

$$\tfrac{1}{2}(\rho u dy)u^2 - \tfrac{1}{2}(\rho u dy)u^2$$
$$= \tfrac{1}{2}\rho u\left(U^2 - u^2\right)dy \text{ -----------------} (i)$$

$\therefore$ Total loss of K.E of fluid = $\int_0^\delta \tfrac{1}{2}\rho u\left(U^2 - u^2\right)dy$

Let ∂_e = Distance by which the plate is displaced to compensate for the reduction in K.E

Then loss of K.E. through ∂_e of fluid flowing with velocity

$$U = \tfrac{1}{2}(\rho U \delta_e)U^2 \text{ ------------} (ii)$$

Equating eqns (i) and (ii), we have

$$\tfrac{1}{2}(\rho u dy)u^2 = \int_0^\delta \tfrac{1}{2}\rho u\left(U^2 - u^2\right)dy$$
$$\delta_e = \frac{1}{U^3}\int_0^\delta u\left(U^2 - u^2\right)dy$$

or

$$\int_0^\delta \frac{u}{U}\left(1 - \frac{u^2}{U^2}\right)dy$$

Momentum Equation for Boundary Layer by Von Karman

Von Karman suggested a method based on the momentum equation by the use of which the growth of a boundary layer along a flat plate, the wall shear stress and the drag force could be determined (when the velocity distribution in the boundary layer is known). Starting from the beginning of the plate, the method can be wed for both laminar and turbulent boundary layers.

The figure below shows a fluid flowing over a thin plate (placed at zero incidence) with a free stream velocity equal to U. Consider a small length dx of the plate at a distance x from the leading edge as shown in fig. (a). Consider unit width of plate perpendicular to the direction of flow.

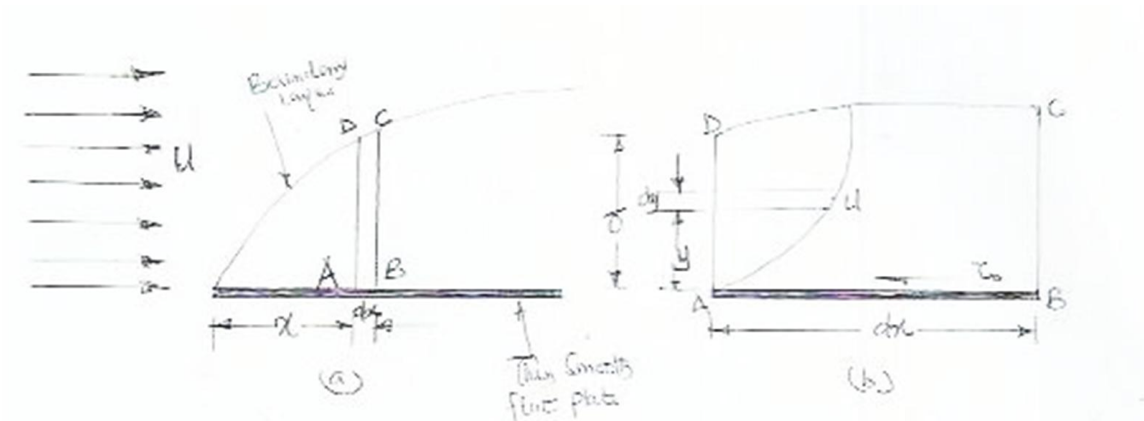


Fig.(a) and (b) Momentum equation for boundary layer by Von Karman

Let ABCD be a small element of a boundary layer (the edge DC represents the outer edge of the boundary layer).

Mass rate of fluid entering through AD

$$=\int_0^\delta \rho u dy+\frac{d}{dx}\left[\int_0^\delta \rho u dy\right]dx --------(ii) \quad \begin{bmatrix} i.e.(mass\ through\ AD)+\frac{d}{dx} \\ (mass\ through\ AD)x\,dx \end{bmatrix}$$

$\therefore$ Mass rate of fluid entering the control volume through the surface DC

= mass rate of fluid through BC – Mass rate of fluid through AD

$$=\int_0^\delta \rho u dy+\frac{d}{dx}\left[\int_0^\delta \rho u dy\right]dx-\int_0^\delta \rho u dy -------(iii)$$

$$=\frac{d}{dx}\left[\int_0^\delta \rho u dy\right]dx -----------------(iv)$$

The fluid is entering through DC with a uniform velocity U.

Momentum rate of fluid entering the control volume of X-direction through AD.

$$\int_0^\delta \rho u^2 dy -----------(v)$$

Momentum rate of fluid leaving the Control Volume in X-direction through BC

$$=\int_0^\delta \rho u^2 dy+\frac{d}{dx}\left[\int_0^\delta \rho u^2 dy\right]dx ----------(vi)$$

Momentum rate of fluid entering the control volume through DC in X-direction

$$= \frac{d}{dx}\left[\int_0^\delta \rho u dy\right] dx \times U \quad (\because Velocity = U) ----------(vii)$$

$$= \frac{d}{dx}\left[\int_0^\delta \rho u U dy\right] dx ------------(viii)$$

$\therefore$ Rate of change of momentum of Control Volume

= Momentum rate of fluid through BC – Momentum rate of fluid through AD – Momentum of fluid through DC

$$= \int_0^\delta \rho u^2 dy + \frac{d}{dx}\left[\int_0^\delta \rho u^2 dy\right] dx - \int_0^\delta \rho u^2 dy + \frac{d}{dx}\left[\int_0^\delta \rho u U dy\right] dx ------(ix)$$

$$= \frac{d}{dx}\left[\int_0^\delta \rho u^2 dy - \int_0^\delta \rho u U dy \;\; dx\right] ------------$$

$$= \frac{d}{dx}\left[\int_0^\delta \left(\rho u^2 dy - \rho u U dy\right)\right] dx$$

$$= \frac{d}{dx}\left[\rho \int_0^\delta \left(u^2 - uU\right) dy\right] dx --------------------(x)$$

As per momentum principle, the rate of change of momentum on the control volume BCD must be equal to the total force on the control volume in the same direction. The only external force acting on the control volume is the share force acting on the side AB in the direction B to A (fig. b) above). The value of this force (drag force) is given by,

$$\Delta F_D = \tau_o \times dx$$

Thus the total external force in the direction of the rate of change of momentum $= -\tau_o \times dx \; -------(xi)$

Equating equation (x) and (xi), we have

$$-\tau_o \times dx = \rho \frac{d}{dx}\left[\int_\delta^\delta \left(u^2 - uU\right) dy\right] dx$$

Or

$$\rho \frac{d}{dx}\left[\int_{\delta}^{\delta}\left(u^2 - uU\right)dy\right]$$

$$or, = \rho \frac{d}{dx}\left[\int_{\delta}^{\delta}\left(uU - u^2\right)dy\right]$$

$$= \rho \frac{d}{dx}\left[\int_0^{\delta} U^2\left(\frac{u}{U} - \frac{u^2}{U^2}\right)dy\right]$$

$$= \rho U \frac{d}{dx}\left[\int_0^{\delta} \frac{u}{U}\left[1 - \frac{u}{U}\right]dy\right]$$

$$or \quad \frac{\tau_o}{\rho U^2} = \frac{d}{dx}\left[\int_0^{\delta} \frac{u}{U}\left(1 - \frac{u}{U}\right)\right]dy \text{ --------- } (xiii)$$

But,

$$\int_0^{\delta} \frac{u}{U}\left(1 - \frac{u}{U}\right)dy = momentum \ thickness \ \theta$$

$$\therefore \frac{\tau_o}{\rho U^2} = \frac{d\theta}{dx} \text{ --------- } (xvii)$$

This equation is known as von Karman momentum equation for boundary layer flow and it is used to find out the frictional drag on smooth flat plate for both laminar and turbulent boundary layer.

The following boundary conditions must be satisfied for any assumed velocity distribution.

(i) At the surface of the plate $y = 0,\ U = 0, \frac{du}{dy} = finite\ value$

(ii) At the outer edge of boundary layer $y = \delta, u = U,\ y = \delta, \frac{du}{dy} = 0$

The sheer stress, τ_o for a given velocity profile in laminar, transition or turbulent zone is obtained from equations (xii) and (xiii) above. Then drag force on a small distance dx of a plate is given by

$$\Delta F_D = shear\ stress \times area$$

$$= \tau_o \times (B \times dx) = \tau_o \times B \times dx \ \left[assuming\ width\ of\ plate\ as\ unity\right]$$

$$where,\ B = width\ of\ the\ plate$$

$$\therefore Total\ drag\ on\ the\ plate\ of\ length\ L\ one\ side,$$

$$F_D = \int \Delta F_D = \int_0^L \tau_o \times B \times dx$$

- The ratio of the shear stress to the quantity $\frac{1}{2}\ell u^2$ is known as the Local co-efficient of drag" (or co-efficient of skin fraction) and is denoted by C_D^* i.e.
  $C_D^* = \dfrac{\tau_o}{\frac{1}{2}\rho u^2}$
- The ratio of the total drag force to the quantity $\frac{1}{2}\ell u^2$ is called 'Average-coefficient of drag' and is denoted by C_D i.e. $C_D^* = \dfrac{F_D}{\frac{1}{2}\rho A U^2}$

ℓ =Mass density of fluid

A = Area of surface/plate, and

U = free stream velocity

EXAMPLE 1

The velocity distribution in the boundary layer is given by $\dfrac{u}{U} = \dfrac{y}{\sigma}$, where u is the velocity y from the plate and u=U at , $y = \delta$, δ being boundary layer thickness. Find

i. The displacement thickness
ii. The momentum thickness
iii. The energy thickness and
iv. The value of $\dfrac{\delta *}{\theta}$

Solution:

Velocity distribution: $\dfrac{u}{U} = \dfrac{y}{\sigma}$

(i) The displacement thickness $\delta *$

$$\delta * = \int_0^\delta \left(1 - \frac{u}{U}\right) dy$$

$$= \int_0^\delta \left(1 - \frac{y}{\delta}\right) dy$$

$$= \left[y - \frac{y^2}{2\delta} \right]_0^\delta$$

$$= \left(\delta - \frac{\delta^2}{2\delta} \right) = \delta - \frac{\delta}{2}$$

$$= \frac{\delta}{2}$$

(ii) The momentum thickness

$$\theta = \int_o^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy$$

$$= \int_o^\delta \frac{y}{\delta}\left(1-\frac{y}{\delta}\right)dy$$

$$= \int_o^\delta \left(\frac{y}{\delta}-\frac{y^2}{\delta^2}\right)dy$$

or

$$\theta = \left[\frac{y^2}{2\delta}-\frac{y^3}{3\delta^2}\right]_0^\sigma = \frac{\delta^2}{2\delta}-\frac{\delta^3}{3\delta^2} = \frac{\delta}{2}-\frac{\delta}{3} = \frac{\delta}{6}$$

(iii)

$$\delta_e = \int_o^\delta \frac{u}{U}\left(1-\frac{u^2}{U^2}\right)dy$$

$$= \int_o^\delta \frac{y}{\delta}\left(1-\frac{y^2}{\delta^2}\right)dy = \int_o^\delta \left(\frac{y}{\delta}-\frac{y^3}{\delta^3}\right)dy$$

$$= \left[\frac{y^2}{2\delta}-\frac{y^4}{4\delta^3}\right]_0^\delta = \frac{\delta^2}{2\delta}-\frac{\delta^4}{4\delta^3}$$

$$= \frac{\delta}{2}-\frac{\delta}{4}$$

$$= \frac{\delta}{4}$$

(iv) The value of $\frac{\delta^*}{\theta}$

$$\frac{\delta^*}{\theta} = \frac{\delta/2}{\delta/6}$$

$$= 3.0$$

Example 2

The velocity distribution in the boundary layer is given by $\frac{u}{U} = \frac{3}{2}\frac{y}{\sigma} - \frac{1}{2}\frac{y^2}{\sigma^2}$, σ being the boundary layer thickness

Calculate the following

(i) The ratio of displacement thickness to boundary layer thickness $\left(\dfrac{\delta *}{\delta}\right)$

(ii) The ratio of momentum thickness to boundary layer thickness $\left(\dfrac{\theta}{\delta}\right)$

Solution

Velocity distribution: $\dfrac{u}{U} = \dfrac{3}{2}\dfrac{y}{\sigma} - \dfrac{1}{2}\dfrac{y^2}{\sigma^2}$

(i) $\dfrac{\delta *}{\delta}$:

$$\delta * = \int_0^\delta \left(1 - \frac{u}{U}\right)dy = \int_0^\delta \left(1 - \frac{3}{2}\frac{y}{\sigma} + \frac{1}{2}\frac{y^2}{\sigma^2}\right)dy$$

$$= \left[y - \frac{3}{2} \times \frac{y^2}{3\sigma} + \frac{1}{2} \times \frac{y^3}{3\sigma^2} \right]_0^\delta$$

$$\left[\sigma - \frac{3}{4}\frac{\sigma^2}{\sigma} + \frac{1}{2}\ \frac{\sigma^2}{3\sigma^2} \right]$$

$$= \left(\sigma - \frac{3}{4}\sigma + \frac{\sigma}{6} \right)$$

$$\sigma * = \frac{5}{12}\sigma$$

$$\therefore \frac{\sigma *}{\sigma} = \frac{5}{12}\sigma.$$

(iii) θ/σ

$$\theta = \int_0^\sigma \frac{u}{U}\left(1-\frac{u}{U}\right)dy$$

$$= \int_0^\sigma \left(\frac{3}{2}\frac{y}{\sigma} - \frac{1}{2}\frac{y^2}{\sigma^2}\right)\left(1-\frac{3}{2}\frac{y}{\sigma}+\frac{1}{2}\frac{y^2}{\sigma^2}\right)dy$$

$$= \int_0^\sigma \left(\frac{3}{2}\frac{y}{\sigma} - \frac{9}{4}\frac{y^2}{\sigma^2}+\frac{3}{4}\frac{y^3}{\sigma^3} - \frac{1}{2}\frac{y^2}{\sigma^2}+\frac{3}{4}\frac{y^3}{\sigma^3} - \frac{1}{4}\frac{y^4}{\sigma^4}\right)dy$$

$$= \int_0^\sigma 3/2\, \frac{y}{\sigma} - \left(\frac{9}{4}\frac{y^2}{\sigma^2}+\frac{1}{2}\frac{y^2}{\sigma^2}\right)+\left(\frac{3}{4}\frac{y^3}{\sigma^3}+\frac{3}{4}\frac{y^3}{\sigma^3}\right) - \frac{1}{4}\frac{y^4}{\sigma^4}\Bigg]dy$$

$$= \int_0^\sigma \left(\frac{3}{2}\frac{y}{\sigma} - \frac{11}{4}\frac{y^2}{\sigma^2}+\frac{3}{4}\frac{y^3}{\sigma^3} - \frac{1}{4}\frac{y^4}{\sigma^4}\right)dy$$

$$= \left[\frac{3}{2}\frac{y^2}{2\sigma} - \frac{11}{4}\frac{y^3}{3\sigma^2}+\frac{3}{2}\times\frac{y^4}{4\sigma^3} - \frac{1}{4}\times\frac{y^5}{4\sigma^4}\right]_0^\delta$$

$$= \left[\frac{3}{2}\times\frac{y^2}{2\sigma}\times\frac{11}{4}\times\frac{y^3}{3\sigma^2}+\frac{3}{2}\times\frac{\sigma^4}{4\sigma^3} - \frac{1}{4}\times\frac{\sigma^5}{5\sigma^4}\right]_0^\delta$$

$$\theta = \left(\frac{3}{4}\sigma - \frac{11}{12}\sigma + \frac{3}{8}\sigma - \frac{1}{20}\sigma\right) = \frac{19}{120}\sigma$$

$$or\ \frac{\theta}{\sigma} = \frac{19}{120}$$

Assignment

(1) If velocity distribution in laminar boundary layer over a flat plate is given by second order polynomial $U = a + by + cy^2$, determine its form using the necessary boundary conditions

(2) The velocity distribution in the boundary layer is given by $\frac{u}{U} = \left(\frac{y}{\sigma}\right)^{\frac{1}{7}}$, calculate the following

(i) Displacement thickness
(ii) Momentum thickness
(iii) Shape factor
(iv) Energy thickness and
(v) Energy loss due to boundary layer if at a particular section, the boundary layer thickness is 25mm and the free stream velocity is 15m/s. If the discharge through the boundary layer region is $6m^3/s$ per metre width, express this energy loss in terms of metres of head. Take $\ell = 1.2 kg/m^3$

(3) In the boundary layer over the face of a high spillway, the velocity distribution was observed to have the following form:

$$\frac{u}{U} = \left(\frac{y}{\sigma}\right)^{0.22}$$

The free stream velocity U at a certain section was observed to be 30m/s and boundary layer thickness of 60mm was estimated from the velocity distribution measured at the section. The discharge passing over the spillway was 6m^3/s per metre length of spillway, calculate

i. The displacement thickness
ii. The energy thickness, and
iii. The loss of energy up to the section under consideration.

Laminar Boundary Layer

Let us find out boundary layer thickness (σ), shear stress (τ_o) local co-efficient of drag (C_D) for the following velocity distribution in the boundary layer:

1. $$\frac{u}{U} = 2\left(\frac{y}{\sigma}\right) - \left(\frac{y}{\sigma}\right)^2$$

2. $$\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\sigma}\right) - \tfrac{1}{2}\left(\frac{y}{\sigma}\right)^3$$

3. $$\frac{u}{U} = 2\left(\frac{y}{\sigma}\right) - 2\left(\frac{y}{\sigma}\right)^3 + \left(\frac{y}{\sigma}\right)^4$$

4. $$\frac{u}{U} = Sin\left(\frac{\pi}{2}\frac{y}{\sigma}\right)$$

Case 1: Velocity distribution: $\frac{u}{U} = 2\left(\frac{y}{\sigma}\right) - \left(\frac{y}{\sigma}\right)^2$ ------- (i)

(i) Boundary layer thickness

We know, $$\frac{\tau_o}{\rho U^2} = \frac{d}{dx}\left[\int_0^{\sigma} \frac{u}{U}\left(1 - \frac{u}{U}\right)dy\right]$$

Substituting the value of $\frac{u}{U}$, we get

$$\frac{\tau_o}{\rho U^2} = \frac{d}{dx}\left[\int_0^{\sigma}\left(\frac{2y}{\sigma} - \frac{y^2}{\sigma^2}\right)\left(1-\left(\frac{2y}{\sigma} - \frac{y^2}{\sigma^2}\right)\right)dy\right]$$

$$= \frac{d}{dx}\left[\int_0^{\sigma}\left(\frac{2y}{\sigma} - \frac{y^2}{\sigma^2}\right)\left(1-\frac{2y}{\sigma} + \frac{y^2}{\sigma^2}\right)dy\right]$$

$$= \frac{d}{dx}\left[\int_0^{\sigma}\left(\frac{2y}{\sigma} - \frac{4y^2}{\sigma^2} + \frac{2y^3}{\sigma^3} - \frac{y^2}{\sigma^2} + \frac{2y^3}{\sigma^3} - \frac{y^4}{\sigma^2}\right)dy\right]$$

$$= \frac{d}{dx}\left[\int_0^{\sigma}\left(\frac{2y}{\sigma} - \frac{5y^2}{\sigma^2} + \frac{4y^3}{\sigma^3} - \frac{y^4}{\sigma^4}\right)dy\right]$$

$$= \frac{d}{dx}\left[\frac{2}{2}\ \frac{y^2}{\sigma} - \frac{5}{3}\frac{y^2}{\sigma^2} + \frac{4}{4}\frac{y^4}{\sigma^3} - \frac{1}{5}\frac{y^5}{\sigma^4}\right]_0^{\sigma}$$

$$= \frac{d}{dx}\left[\sigma - \frac{5}{3}\sigma + \sigma\frac{1}{5}\sigma\right] = \frac{d}{dx}\left(\frac{2}{15}\sigma\right)$$

$$\therefore \tau_o = \rho U^2 \times \frac{d}{dx}\left(\frac{2}{15}\sigma\right) = \frac{2}{15}\rho U^2\ \frac{d\delta}{dx} ----------(ii)$$

Also, according to Newton's law of viscosity

$$\tau_o = \mu\ \left(\frac{dy}{dx}\right)_{y=0} ------------(iii)$$

$$But \quad u = U\ \left(\frac{2y}{\sigma} - \frac{y^2}{\sigma^2}\right)$$

$$and \quad \frac{du}{dx} = U\ \left(\frac{2}{\sigma} - \frac{2y}{\sigma^2}\right),\ U\ \ being\ \ cons\tan t$$

$$\therefore\ \left(\frac{du}{dx}\right)_{y=0} = U\left(\frac{2}{\sigma} - 0\right) = \frac{2U}{8}$$

Substituting this value in (iii), we get

$$\tau_o = \frac{2\mu U}{\sigma} -----------(iv)$$

Equating the values of τ_o given by equations (ii) and iv, we get

$$\frac{2}{15}\rho U^2 \frac{d\sigma}{dx} = \frac{2\mu U}{\sigma}$$

$$or \quad \sigma.\frac{d\sigma}{dx} = \frac{15\mu U}{\rho U^2} = \frac{15\mu}{\rho U^2}$$

$$or \ \sigma.d\sigma = \frac{15\mu}{\rho U} dx$$

Integrating both sides, we get

$$\frac{\delta^2}{2} = \frac{15\mu}{\rho U} x + c \quad (where\ C = Cons\tan t\ of\ \text{int}\, egration)$$

$$At\ x = 0,\ \delta = 0 \therefore C = 0$$

$$\therefore\ \frac{\delta^2}{2} = \frac{15\mu}{\rho U}$$

$$or\ \delta = \sqrt{\frac{2 \times 15\mu x}{\rho U}} = 5.48\sqrt{\frac{\mu x}{\rho U}}$$

$$= 5.48\sqrt{\frac{\mu x \times x}{\rho U \ \times x}} = 5.48 \ \sqrt{\frac{x^2}{\text{Re}_x}}$$

$$\left(where\ \text{Re}_x = \frac{\rho U x}{\varpi},\right)$$

$$or\ \sigma = 5.48 = \frac{x}{\sqrt{\text{Re}_x}} ------(v)$$

(ii) Shear stress τ_o :

From equation (iv), we have

$$\tau_o = \frac{2\mu U}{\sigma}$$

But $\sigma = 5.48 \dfrac{x}{\sqrt{\text{Re}_x}}$

$$\therefore \tau_o = \frac{2\mu U}{5.48 \dfrac{x}{\sqrt{\text{Re}_x}}} = \frac{2\mu U \sqrt{\text{Re}_x}}{5.48\, x} = 0.365 \frac{\mu U}{x} \sqrt{\text{Re}_x} -------------(vi)$$

(iii) Local Co-efficient of drag, C_D^*

$$\tau_o = \frac{0.365\,\mu U}{x} = \sqrt{\mathrm{Re}_x}$$

$$also\ \tau_o = C_D^* \frac{\rho U^2}{2} ----------- (vii) \left(where\ C_D^* = local\ coefficient\ of\ drag \right)$$

Equating the two of τ_o, given by equation (vi) and (vii), we get

$$C_D^* = \frac{0.365\mu U}{x}\sqrt{\mathrm{Re}_x} \quad or \quad C_D^* = 0.365 \times 2 \times \frac{\sqrt{\mathrm{Re}_x}}{\frac{\rho U x}{\mu}}$$

$$= \frac{0.73}{\sqrt{\mathrm{Re}_x}}$$

(iv) Co-efficient of drag, C_D:

We know that $C_D = \dfrac{F_D}{\frac{1}{2}\rho A U^2}$

Where, $F_D = \int_0^L \tau_o \times B \times dx$

$$= \int_0^l 0.365\ \frac{\mu U}{x}\ \sqrt{\mathrm{Re}_x}\ \times B \times\ dx$$

$$= 0.365 \int_0^l\ \frac{\mu U}{x}\ \sqrt{\frac{\rho U x}{\mu}}\ \times B \times\ dx \left(\because \mathrm{Re}_x = \frac{\rho U x}{\mu} \right)$$

$$= 0.365\ \int_0^l\ \frac{\mu U}{x}\ \sqrt{\frac{\rho U x}{\mu}}\ \times \frac{1}{\sqrt{x}}\ \times B \times\ dx$$

$$= 0.365\ \mu U\ \sqrt{\frac{\rho U}{\mu}} \times B \int_0^l x^{-\frac{1}{2}}\ \times\ dx$$

$$= 0.365\ \mu U\ \sqrt{\frac{\rho U}{\mu}} \times B \left[\frac{x^{-\frac{1}{2}}}{\frac{1}{2}} \right]$$

$$= 0.365 \times 2 \mu U B\ \sqrt{\frac{\rho U}{\mu}} \times\ B\sqrt{L}$$

$$Or\ \ F_D = 0.73\ \ \mu U B\ \sqrt{\frac{\rho U}{\mu}}$$

$$\therefore C_D\ = \frac{0.73 \mu U B \sqrt{\frac{\rho U}{\mu}}}{\frac{1}{2}\ \rho A U^2}$$

(Where A – area of plate = L x B, L and B being length and width of the plate respectively)

$$\therefore C_D\ = \frac{0.73 \mu U B \sqrt{\frac{\ell U L}{\mu}}}{\frac{1}{2} \ell \times L \times B \times U^2} = \frac{1.46 \mu}{\ell L.U}\ \sqrt{\frac{\ell U L}{\mu}}$$

$$= \frac{1.46 \sqrt{\mu}}{\sqrt{\ell L.U}} = 1.46 \sqrt{\frac{\mu}{\ell L.U}}\ \ = \frac{1.46}{\sqrt{\mathrm{Re}_l}}$$

CASE 2: Velocity distribution: $\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\sigma}\right) - \frac{1}{2}\ \left(\frac{y}{\sigma}\right)^3$

i. Boundary layer thickness δ:

$$\frac{\tau_o}{\ell U^2} = \frac{d}{dx}\left[\int_0^l \frac{u}{U}\left(1 - \frac{u}{U}\right) dy \right] \ldots\ldots\ldots\ldots\ldots\ldots.(i)$$

Substituting the value of $\frac{u}{U}$, we get

$$\frac{\tau_o}{\ell U^2} = \frac{d}{dx}\left[\int_0^{\sigma}\left(\frac{3}{2}\frac{y}{\sigma} - \frac{1}{2}\frac{y^3}{\sigma^3}\right)\left(1-\frac{3}{2}\frac{y}{\sigma} + \frac{1}{2}\frac{y^3}{\sigma^3}\right)\right]$$

$$= \frac{d}{dx}\left[\int_0^{\sigma}\left(\frac{3}{2}\frac{y}{\sigma} - \frac{9}{4}\frac{y^2}{\sigma^2} + \frac{3}{4}\frac{y^4}{\sigma^4} - \frac{1}{2}\frac{y^3}{\sigma^3} + \frac{3}{4}\frac{y^4}{\sigma^4} - \frac{1}{4}\frac{y^6}{\sigma^6}\right)dy\right]$$

$$= \frac{d}{dx}\left[\frac{3}{2}\times\frac{1}{2}\frac{y^2}{\sigma} - \frac{9}{4}\times\frac{1}{3}\frac{y^3}{\sigma^2} + \frac{3}{4}\times\frac{1}{5}\frac{y^5}{\sigma^4} - \frac{1}{2}\times\frac{1}{4}\frac{y^4}{\sigma^3} + \frac{3}{4}\frac{y^5}{\sigma^4}\times\frac{1}{7}\frac{y^7}{\sigma^6}\right]$$

$$\frac{d}{dx}\left[\frac{3}{4}\delta - \frac{3}{4}\delta + \frac{3}{20}\delta - \frac{1}{8}\delta - \frac{1}{28}\delta\right]$$

$$= \frac{39}{280}\frac{d\delta}{dx}$$

$$or\ \tau_o = \ell U^2 \times \frac{39}{280}\frac{d\delta}{dx}$$

$$= \frac{39}{280}\ell U^2\frac{d\delta}{dx} ----------------(ii)$$

Also

$$\tau_o = \mu\left(\frac{du}{dy}\right)_{y=0}$$

But

$$u = U\left[\frac{3}{2}\left(\frac{y}{\sigma}\right) - \frac{1}{2}\left(\frac{y}{\sigma}\right)^3\right]$$

And

$$\frac{du}{dy} = U\left(\frac{3}{2\sigma} - \frac{3}{2}\frac{y^2}{\sigma^3}\right)$$

$$\tau_o = \mu\left(\frac{du}{dy}\right)_{y=0} = \mu U\left(\frac{3}{2\sigma} - 0\right) = \frac{3\mu U}{2\sigma} --------(iii)$$

Equating the two values of τ_o given by equation (ii) and (iii), we get

$$\frac{39}{280}\ell U^2\frac{d\delta}{dx} = \frac{3\mu U}{2\delta}$$

$$\delta.d\delta = \frac{3}{2}\mu U \times \frac{280}{39} \times \frac{dx}{\rho U^2}$$

Integrating both sides, we get

$$\delta^2 = \frac{420}{39}\frac{\mu}{\ell U}x + c$$

(Where C=constant of integration)

When x=0, $\delta = 0 \therefore C = 0$

$$\therefore \frac{\delta^2}{2} = \frac{420}{39}\frac{\mu}{\rho u}X$$

Or

$$\delta = \sqrt{\frac{420\ x\ 2}{39} \cdot \frac{\mu}{\rho u}}$$

$$= 4.64\sqrt{\frac{\mu x}{\rho u}}$$

$$= 4.64\sqrt{\frac{\mu x}{\rho u} \times \frac{x}{x}}$$

$$= 4.64\sqrt{\frac{\mu x}{\rho u}\ .x}$$

$$= \frac{4.64x}{\sqrt{\mathrm{Re}_x}}$$

Shear Stress, τ_o

$$\tau_o = \frac{3\mu U}{2\delta}$$

$$But\ \delta = \frac{4.64x}{\sqrt{\mathrm{Re}_x}}$$

$$\therefore \tau_o = \frac{3\mu U}{2 \times \frac{4.64x}{\sqrt{\mathrm{Re}_x}}} = \frac{3}{9.28}\ \frac{\mu U\sqrt{\mathrm{Re}_x}}{x} \quad -------(v)$$

$$= 0.323\ \frac{\mu U}{x}\sqrt{\mathrm{Re}_x}$$

(iii) Local Coefficient of Drag, C_D^* :

$$\tau_o = 0.323 \frac{\mu U}{x} \sqrt{\mathrm{Re}_x}$$

Also,

$$\tau_o = C_D^* \frac{\ell U^2}{2}$$

$$\therefore \; C_D^* \frac{\ell U^2}{2} = 0.323 \frac{\mu U}{x} \sqrt{\mathrm{Re}_x}$$

$$or \;\; C_D^* \;\; \frac{0.646}{\sqrt{\mathrm{Re}_x}}$$

(iv) Co-efficient of drag (C_D):

$$C_D = \frac{F_D}{\frac{1}{2}\rho A U^2}$$

$$where, \; F_D = \int_0^L \tau_o \times B \times dx$$

$$= \int_0^L \frac{\mu U}{x} \mathrm{Re}_x \;\; \times B \times dx$$

$$= 0.323 \int_0^L \frac{\mu U}{x} \sqrt{\frac{\rho U x}{\mu}} \;\; \times B \times dx$$

$$= 0.323 \mu U \sqrt{\frac{\rho U x}{\mu}} \times B \int_0^L x^{-\frac{1}{2}}\, dx = 0.323 \mu U \sqrt{\frac{\rho U}{\mu}} \;\; \times B \left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}} \right]_0^L$$

$$= 0.323 \mu U \sqrt{\frac{\rho U}{\mu}} \times B \int_0^L x^{-\frac{1}{2}} \, dx$$

$$= 0.323 \mu U \sqrt{\frac{\rho U}{\mu}} \times B \left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}} \right]_0^L$$

$$= 0.323 \times 2 \mu U \sqrt{\frac{\rho U}{\mu}} \times B \times \sqrt{L}$$

$$or \; F_D = 0.646 \mu U \sqrt{\frac{\rho U L}{\mu}} \times B \; -------------- (vii)$$

$$\therefore C_D = \frac{0.646 \mu U B \sqrt{\frac{\rho U L}{\mu}}}{\frac{1}{2} \ell \times L \times B \times U^2}$$

$$= 0.646 \times 2 \times \frac{\mu}{\rho U L} \times \sqrt{\frac{\rho U L}{\mu}}$$

$$= \frac{1.292}{\sqrt{\frac{\rho U L}{\mu}}}$$

$$Or, \; C_D = \frac{1.292}{\sqrt{\mathrm{Re}_L}} \left(where, \; \mathrm{Re}_L = \sqrt{\frac{\rho U L}{\mu}} \right) ---- (vii)$$

CASE 3: Velocity Distribution:

$$F_D = 0.636 \; \mu U B \sqrt{\frac{\ell U L}{\mu}}$$

$$\therefore C_D = \frac{0.686 \mu U B \sqrt{\frac{\ell U L}{\mu}}}{\frac{1}{2} \ell A U^2}$$

$$= \frac{0.686 \mu U B \sqrt{\frac{\ell U L}{\mu}}}{\frac{1}{2} \ell \times L \times B \times U^2}$$

$$(where \; A = L \times B)$$

$$= 0.686 \times 2 \times \frac{\mu}{\ell U L} \times \sqrt{\frac{\ell U L}{\mu}}$$

$$=1.372\frac{1}{\sqrt{\frac{\ell UL}{\mu}}}$$

$$C_D=\frac{1.372}{\sqrt{\text{Re}_L}}------------------(xi)$$

CASE 4: Velocity distribution $\frac{u}{U}Sin\left(\frac{\pi}{2}\frac{y}{\sigma}\right)$.

(i) $\delta=\frac{4.795x}{\sqrt{\text{Re}_x}}$

(ii) $\tau_o=0.327\frac{\mu U}{x}\sqrt{\text{Re}_x}$

(iii) $C_D^*=\frac{0.654}{\sqrt{\text{Re}_x}}$

(iv) $C_D=\frac{1.31}{\sqrt{\text{Re}_L}}$

Example 1

Air at atmospheric pressure and at 400K flows over a flat plate with a velocity of 5m/s. The transition from laminar to turbulent flow is assumed to take place at a Reynold number of 5 x 10^5; determine the distance from the leading edge of the plate at which transition occurs.

Solution

At $T_\infty=400K$ and at atmospheric pressure, from tables of properties of air,

$$\rho_a = 0.8826kg/m^3,\ \mu=2.286\times10^{-5}\ kg/m.s$$
$$\nu = 25.90\times 10^{-6}\ m^2/s,\ \text{Pr}= 0.689$$

The transition occurred at a distance L from the leading edge.

$$\text{Re}_L = \frac{U_\infty L}{\nu} = \frac{5\times L}{2.59\times10^{-5}} = 5\times10^5$$
$$L = 2.59m$$

Example 2

Air at atmospheric press and at 350k flows over a flat plate with a velocity of 5m/s. The average drag coefficient Cm over a distance of 2m from the leading edge is 0.0019. Calculate the drag force acting per 1m width of the plate over the distance of 2m from the leading edge.

Solution

From

$$C_D = \frac{F_D}{\frac{1}{2}\ell A U^2}$$

$$F_D = C_D \times \tfrac{1}{2}\ell A U^2$$

$$= WL \;\; C_D \frac{\rho U_\infty^2}{2}$$

At T_∞ of 350k and at atmospheric pressure

$$\rho_a = 0.9980\, Kg/m^3,\ \mu = 2.075 \times 10^{-5}\ \ Kg/m.s.$$

$$\nu = 20.76 \times 10^{-6} m^2/s$$

$$F_D = (1)(2)(0.0019)\frac{(0.9980)(5)^2}{2}$$

$$= 0.0019\ (0.9980)\ (25)$$

$$= 0.0474 N$$

$$\text{Re} = \frac{U_\infty L}{\nu} = \frac{5 \times 2}{20.7 \times 10^{-6}} = \frac{10}{20.76} \times 10^6$$

$$= 4.8 \times 10^5$$

The flow is Laminar.

Example 3

Oil with a free stream velocity of 3.0m/s flows over a thin plate 1.25m wide and 2m long. Determine the boundary layer thickness and the shear stress at mid-length and calculate the total, double-sided resistance of the plate ($\rho = 860 kgm^{-3}$, $\nu = 10^{-5} m^2/s$, $\nu = \mu/\rho$)

Solution

Given: $U_s = 3.0 m/s,\ width = 1.25m,\ L = 2m.\ \ell = 860 kg/m^3$

$$\delta = (U - 0.99U_s)$$
$$U = 0.99 \times 3$$
$$= 2.97m$$

Calculate the Reynolds number at x=1m

$$\mathrm{Re}_x = \frac{U_s x}{\nu} = \frac{3 \times 1}{10^{-5}} = 300000 = 3 \times 10^5$$
$$\therefore \mathrm{Re}_x^{\frac{1}{2}} = 547.7$$
$$= 5.48 \times 10^2$$

Note that Re is low enough to allow the laminar boundary layer to survive over the whole plate.

From

$$\tau_o = 0.323 \frac{\mu U}{x} \mathrm{Re}_x{}^{\frac{1}{2}}$$
$$= 0.323 \times 10^{-5} \times 860 \times \frac{3}{1} \times 5.48 \times 10^2$$
$$= 4.57 N/m^2$$
$$\left[Note: \nu = {}^{\mu}\!/_{\rho}, \mu = \nu\rho \right]$$

The skin friction coefficient (Coefficient of drag) is given by (C_D)

$$C_D = \frac{F_D}{\frac{1}{2}\rho U^2 \, L \times B} = \frac{F_D}{\frac{1}{2}\rho U_3^2 A}$$

$$F_D \; (skin \;\; friction \;\; force) = C_D \times \tfrac{1}{2}\rho U_S^2 \; l \times b \;\; (one \;\; side \;\; resistance)$$

For double sided

$F_D = 2 \times \frac{1}{2} \ell U_3^2 \times l \times b \times C_D$

$= 2 \times \frac{1}{2} \times 860 \times 3^2 \times 2 \times 1.25 \times 1.292 \ \mathrm{Re}_l^{\frac{1}{2}}$

Note $\mathrm{Re}_l \ at \ x = 2m = 6 \times 10^5$

$\therefore F_D = 2 \times \frac{1}{2} \times 860 \times 3^2 \times 2 \times 1.25 \times \dfrac{1.292}{\left(6 \times 10^5\right)^{\frac{1}{2}}}$

$= 860 \times 18 \times 1.25 \times 1.67 \times 10^{-3}$

$= 32.2755N$

$\approx 32.3N$

Example 4

Air at $\dfrac{1}{20}$ atm and at 345K has and $\mu = 2.052 \times 10^{-5} Kg/ms$. Calculate the prandtl number.

Solution

$$\mathrm{Pr} = \frac{\nu}{\alpha} = \frac{\mu/\rho}{k/\rho C_P}$$

$$\nu = \frac{2.052 \times 10^{-5}}{0.0508} = \frac{0.2052}{508} = 4.0394 \times 10^{-4} m^2/s$$

$$\alpha = \frac{0.05}{0.0508 \times 1009} = \frac{5}{5.08 \times 1009}$$

$$= 9.7547 \times 10^{-4} m^2/s$$

$$\therefore \mathrm{Pr} = \frac{4.0394 \times 10^{-4}}{9.7547 \times 10^{-4}}$$

$= 0.414$

Turbulent Boundary Layer (TBL)

Turbulent flow

Fluid motion is highly irregular, and is characterized by velocity fluctuations. These fluctuations enhance the transfer of momentum, energy and species, and hence increase surface friction as well as convection transfer rates. Fluid mixing resulting from the fluctuations makes turbulent B.L thickness larger and BL profiles (velocity, temp and conc.) flatter than in laminar flow. In the TBL, 3 different regions may be delineated

(a) Laminar or viscous sublayer – in which transport is dominated by diffusion and the velocity profile is nearly linear.
(b) Buffer layer – adjacent layer to viscous sublayer in which diffusion and turbulent mixing are comparable.
(c) Turbulent zone – transport is dominated by turbulent mixing

The location x_c at which transition begins is determined by a dimensionless grouping of variables called Reynolds numbers

$$\mathrm{Re}_x = \frac{\rho U_\infty X}{\mu}$$

$\mathrm{Re}_{x,c}$ for BL calculation is taken to be 5 x 10^5

For a flow over a flat plate, the value of $\mathrm{Re}_{x,c}$ varies from 1 x 10^5 to 3 x 10^6 depending on surface roughness and the turbulence level of the free stream.

x in the above expression is the characteristic length, the distance measured along the plate.

Turbulent Boundary Layer

As compared to laminar boundary layers, the turbulent boundary layers are thicker. For in a turbulent boundary layer, the velocity distribution is more uniform than in a laminar boundary layer due to intermingling of fluid particles between different layers of the fluid. The velocity distribution in a turbulent boundary layer follows a logarithmic law i.e. u~log y, which can also be represented by a power law of the type.

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^n \text{-------------}(i)$$

Where, n = $\frac{1}{7}$ (approx..) for Re < 10^7 but > 5 x 10^5

$$\therefore \frac{u}{U} = \left(\frac{y}{\sigma}\right)^{\frac{1}{7}} \text{-------------------}(ii)$$

This is known as one-seventh power law

Let us now find the value of $\delta, \tau_o, C_D^* \ F_D, C_D$ for the velocity distribution given by equation (ii) i.e. $\frac{u}{U} = \left(\frac{y}{\sigma}\right)^{\frac{1}{7}}$

(i) $\sigma = \frac{0.371x}{(\mathrm{Re}_x)^{\frac{1}{5}}}$

(ii) $$\tau_o = \frac{\ell U^2}{2} \times \frac{0.0576}{(\mathrm{Re}_x)^{\frac{1}{5}}}$$

(iii) $$C_D^* = \frac{0.0576}{(\mathrm{Re}_x)^{\frac{1}{5}}}$$

(iv) $$F_D = \frac{\ell U^2}{2} \times \frac{0.072}{(\mathrm{Re}_L)^{\frac{1}{5}}} \times B \times L$$

(v) $$C_D = \frac{0.072}{(\mathrm{Re}_L)^{\frac{1}{5}}}$$

Note: This is valid for $5 \times 10^5 < \mathrm{Re}_L < 10^7$

For Reynolds no between 10^7 and 10^9, the following relationship suggested by Prandtl and Schlichting hold good

$$C_D = \frac{0.455}{(\log_{10} \mathrm{Re}_L)^{2.58}}$$

Example

Air flows over a smooth flat plate at a velocity of 4.39m/s. The density of air is 1.031kg/m^3 and the kinematic viscosity is 1.34 x 10^{-5}m^2/s. The plate's length is 12.2m in the direction of the flow. Calculate

(a) The boundary layer thickness at 15.24cm and 12.2m respectively from the leading edge.
(b) The drag coefficient C_D, for the plate surface

Solution

At the location x = 15.24cm, the Reynolds number is

$$\mathrm{Re}_x = \frac{Ux}{\nu} = \frac{4.39 \times 15.24 \times 10^{-2}}{1.34 \times 10^{-5}} = 5 \times 10^4$$

and the flow is laminar. The boundary layer thickness is obtained from Blasius solution.

$$\delta = \frac{5x}{\sqrt{\mathrm{Re}x}}$$

$$= \frac{5 \times 15.24}{\sqrt{5 \times 104}} = 340.8 \times 10^{-3}\ cm$$

At the location x = 12.2m, the Reynolds number is

$$\mathrm{Re}_x = \frac{4.39 \times 12.2}{1.34 \times 10^{-5}} = 4 \times 10^6$$

And the flow is turbulent. The boundary layer thickness is

$$\sigma = \frac{0.37x}{(\mathrm{Re}_x)^{\frac{1}{5}}} = \frac{0.37 \times 12.2}{(4 \times 10^6)^{\frac{1}{5}}} = 0.216m$$

The drag coefficient C_D can be obtained from

$$C_D = \frac{0.072}{(\mathrm{Re}_L)^{\frac{1}{5}}}$$

$$= \frac{0.072}{(5 \times 10^4)^{\frac{1}{5}}}$$

$$= \qquad (for\ la\min ar\ \ flow)$$

$$C_D = \frac{0.072}{(4 \times 10^6)^{\frac{1}{5}}}$$

$$= \qquad (for\ Turbulent\ \ flow)$$

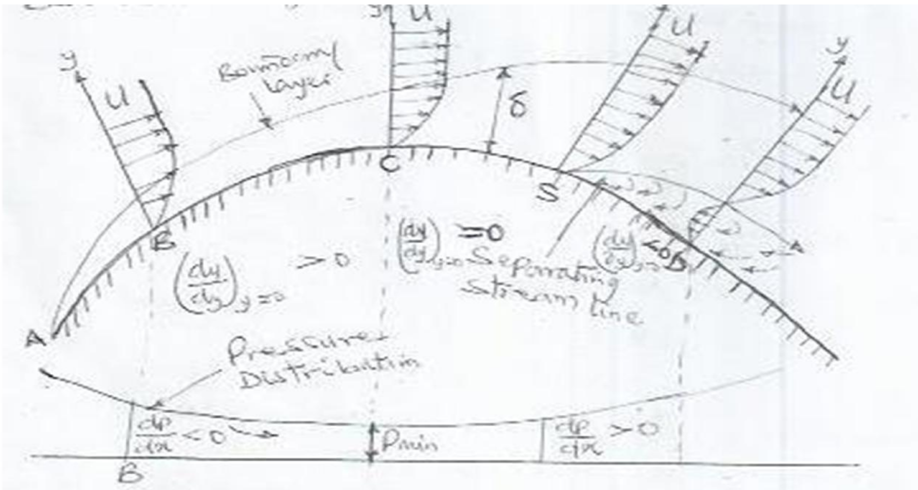
SEPARATION OF BOUNDARY LAYER

When a solid body is immersed in a flowing fluid, a thin layer of fluid called the boundary layer is formed adjacent to the solid body. In this thin layer of fluid, the velocity varies from zero to free stream velocity in the direction normal to the solid body. Along the length of the solid body, the thickness of the boundary layer increases. The fluid layer adjacent to the solid surface has to do work against surface friction at the expense of its kinetic energy. This loss of the kinetic energy is recovered from the immediate fluid layer in contact with the layer adjacent to solid surface through momentum exchange process. Thus the velocity of the layer goes on decreasing. Along the length of the solid body, at a certain point a stage may come when the boundary layer may not be able to keep sticking to the solid body if it cannot provide kinetic energy to overcome the resistance offered by the solid body. In other words, the boundary layer will be separated from the surface. This phenomenon is called the boundary layer separation. The point on the body at which the boundary layer is on the verge of separation from the surface is called point of separation.

Effect of Pressure Gradient on Boundary Layer Separation

The effect of pressure gradient $\left(\frac{dp}{dx}\right)$ on boundary layer separation can be explained by considering the flow over a curved surface ABCSD as shown in the figure below. In the region ABC of the curved surface, the area of flow decreases and hence velocity increases. This means that flow get accelerated in this region. Due to the increase of the velocity, the pressure decreases in the direction of the flow and hence pressure gradient $\frac{dp}{dx}$ is negative in this region. As long as $\frac{dp}{dx}$<0, the entire boundary layer moves forward as shown.

Region CSD of the curved: the pressure is minimum at the points C. Along the region CSD of the curved surface, the area of flow increases and hence velocity of flow along the direction of fluid decreases. Due to decrease of velocity, the pressure increases in the direction of flow and hence pressure gradient $\frac{dp}{dx}$ is positive or $\frac{dp}{dx}$ >0. Thus is the region CSD, the pressure gradient is positive and velocity of fluid layers along the direction of flow decreases. As earlier mentioned, the velocity of the layer adjacent to the solid surface along the length of the solid surface goes on decreasing as the kinetic energy of the layer is used to overcome the frictional resistance of the surface. Thus the combine effect positive pressure gradient and surface resistance reduces the momentum of the fluid. A stage comes, when the momentum of the fluid is unable to overcome the surface resistance and the boundary layer starts separating from the surface at the point S. Downstream the point S, the flow is taking place in reverse direction and the velocity gradient becomes negative.



Effect of pressure gradient on boundary layer separation

The flow separation depends upon factors such as

(i) The curvature of the surface
(ii) The Reynolds number of flow
(iii) The roughness of the surface

The velocity gradient for a given velocity profile, exhibits the following characteristics for the flow to remain attached, get detached or be on the verge of separation:

1 $\left(\frac{du}{dy}\right)_{y=0}$ $is +ve$ ------ attached flow (the flow will not separate)

2 $\left(\frac{du}{dy}\right)_{y=0}$ $is\ zero$ ------ The flow is on the verge of separation

3 $\left(\frac{du}{dy}\right)_{y=0}$ $is -ve$ ------ Separated flow

Methods of preventing the Separation of Boundary Layer

The following are some of the methods generally adopted to retard or arrest the flow separation:

1. Streamlining the body shape
2. Tripping the boundary layer from laminar to turbulent by provision of surface roughness
3. Sucking the retarded flow
4. Injecting high velocity fluid in the boundary layer
5. Providing slots near the leading edge

6. Guidance of flow in a confined passage
7. Providing a rotating cylinder near the leading edge
8. Energizing the flow by introducing optimum amount of swirl in the incoming flow

Example

For the following velocity profiles, determine whether the flow is attached or detached or on the verge of separation:

i. $\dfrac{u}{U} = 2\left(\dfrac{y}{\sigma}\right) - \left(\dfrac{y}{\sigma}\right)^2$ ii. $\dfrac{u}{U} = 2\left(\dfrac{y}{\sigma}\right)^2 + \left(\dfrac{y}{\sigma}\right)^3 + 2\left(\dfrac{y}{\sigma}\right)^4$

iii. $\dfrac{u}{U} = 2\left(\dfrac{y}{\sigma}\right) - \left(\dfrac{y}{\sigma}\right)^3 + 2\left(\dfrac{y}{\sigma}\right)^4$

Solution

i. $\dfrac{u}{U} = 2\left(\dfrac{y}{\sigma}\right) - \left(\dfrac{y}{\sigma}\right)^2 \; or \; U = 2U\left(\dfrac{y}{\sigma}\right) - U\left(\dfrac{y}{\sigma}\right)^2$

Differentiating w.r.t.y the above equation, we get

$$\frac{du}{dy} = 2U\left(\frac{1}{\sigma}\right) - 2U\left(\frac{y}{\sigma}\right) \times \frac{1}{\sigma}$$

$$At \quad y = 0, \left(\frac{du}{dy}\right)_{y=0} = \frac{2U}{\sigma}$$

$$As \left(\frac{du}{dy}\right)_{y=0} \; is + ve, the \; given \; flow \; is \; attached$$

ii. $\dfrac{u}{U} = 2\left(\dfrac{y}{\sigma}\right) + \left(\dfrac{y}{\sigma}\right)^3 + 2\left(\dfrac{y}{\sigma}\right)^4$

or $u = -2U\left(\dfrac{y}{\sigma}\right) + \left(\dfrac{y}{\sigma}\right)^3 + U\left(\dfrac{y}{\sigma}\right)^3 + 2U\left(\dfrac{y}{\sigma}\right)^4$

$$\frac{du}{dy} = 2U\left(\frac{1}{\sigma}\right) - 3U\left(\frac{y}{\sigma}\right)^2 \times \frac{1}{\sigma} + 8U\left(\frac{y}{\sigma}\right)^3 \times \frac{1}{\sigma}$$

$$At \quad y = 0, \left(\frac{du}{dy}\right)_{y=0} - \frac{2U}{\sigma}$$

$$As \left(\frac{du}{dy}\right)_{y=0} \quad is -ve, \, the\ given\ flow\ is\ \mathrm{det}ached \ (i.e. the\ flow has\ separated)$$

iii. $$\frac{u}{U} = \left(\frac{y}{\sigma}\right)^2 + \left(\frac{y}{\sigma}\right)^3 + 2\left(\frac{y}{\sigma}\right)^4$$

$$\text{Or } u = -2U\left(\frac{y}{\sigma}\right)^2 + \left(\frac{y}{\sigma}\right)^3 - 2U\left(\frac{y}{\sigma}\right)^4$$

$$\therefore \frac{du}{dy} = 4U\left(\frac{y}{\sigma}\right) \times \frac{1}{y} + 3U\left(\frac{y}{\sigma}\right) \times \frac{1}{\sigma} - 8U\left(\frac{y}{\sigma}\right)^3 \times \frac{1}{\sigma}$$

$$At \quad y = 0, \left(\frac{du}{dy}\right)_{y=0} = 0$$

$$As \left(\frac{du}{dy}\right)_{y=0} = 0, \ the\ given\ flow\ is\ on\ the verge\ of\ separation$$

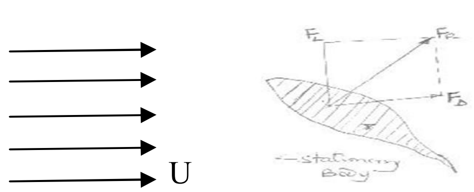
REGIMES OF EXTERNAL FLOW

When a fluid is flowing over a stationary body, a force is exerted by the fluid on the body. Similarly, when a body is moving in a stationary fluid, a force is exerted by the fluid on the body. Also, when both the body and fluid are moving at different velocities, a force is exerted by the fluid on the body. Some of the examples of the fluids flowing over stationary bodies or bodies moving in a stationary fluid are:

(a) Flow of air over buildings,
(b) Flow of water over bridges
(c) Submarines, ships, airplanes and automobiles moving through water and air

Force Exerted by a Flowing fluid on Stationary Bodies

Consider a body held stationary in a real fluid which is flowing at a uniform velocity U as shown in the figure below



Force on a stationary body

The fluid will exert a force on the stationary body. The total force (F_R) exerted by the fluid on the body is perpendicular to the surface of the body. Thus the total force is inclined to the direction of motion.

The total force can be resolved into two components, or in the direction of motion and the other perpendicular to the direction of motion.

DRAG

When a body is immersed in a fluid and is in relative motion with respect to it, the drag is defined as that component of the resultant or total force (F_R) acting on the body which is in the direction of the relative motion. This is denoted by F_D

LIFT

The component of the total or resultant force (F_R) acting in the direction normal or perpendicular to the relative motion is called lift i.e. the force component perpendicular to drag. This component is denoted by F_L. Lift force occurs only when the axis of the body is inclined to the direction of fluid flow. If the axis of the body is parallel to the direction of fluid flow, lift force is zero. In that case only drag force acts. If the fluid is assumed ideal and the body is symmetrical such as a sphere or cylinder, both the drag and lift will be zero.

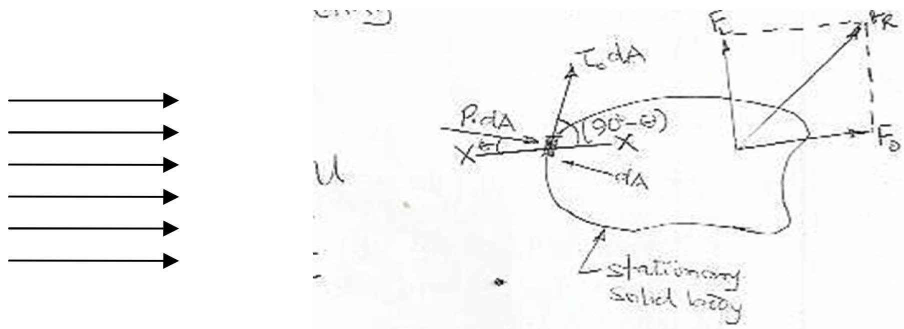
Recall, frictional drag was discussed in connection with the boundary layer theory. It is the force on the body acting in the direction of relative motion due to fluid shear stress at the surface. Thus, in external flow, the immersed body is subjected to frictional drag over its entire surface. Total drag on the body, often called profit drag is therefore made up of two contributions, namely the pressure drag and the skin friction drag. Thus, profile drag = pressure drag + skin frictional drag.

EXPRESSION FOR DRAG AND LIFT

Consider an arbitrary shaped solid body placed in a real fluid, which is flowing with a uniform velocity U in a horizontal direction as shown in the figure below. Consider a small elemental area dA on the surface of the body.

The force acting on the surface area dA are:

1. Pressure force equal to pxdA, acting perpendicular to the surface and
2. Shear force equal to $\tau_o \times dA$, acting along the tangential direction to the surface



Drag and Lift

Let θ = Angle made by pressure force with horizontal direction

(a) Drag force (F_D): The drag force on elemental area = force due to pressure in the direction of fluid motion + force due to shear stress in the direction of fluid motion

$$= PdA\,Cos\,\theta + \tau_o\, dA\,Cos\left(90^\circ - \theta\right) = Pda\ Cos\theta + \tau_o\ dASin\theta$$

$$\therefore Total\ drag,\ F_D = Summation\ of\ PdACos\theta + Summation\ of\ \tau_o dA\,Sin\theta$$

$$= \int PCos\,\theta dA + \int \tau_o\ Sin\theta\, dA - - - - - - - - - - - - - (i)$$

The term $\int PCos\theta dA$ is called the pressure drag or form drag while the term $+\int \tau_o Sin\theta\ dA$ is called the friction drag or skin drag or shear drag.

(b) Lift Force (F_L): The lift force on elemental area = Force due to pressure in the direction perpendicular to the direction of motion + Force due to shear stress in the direction perpendicular to the direction of motion

$$= - PdA\,Sin\,\theta + \tau_o\ dA\,Sin\theta\left(90^o - \theta\right) = - PdA\,Sin\,\theta\ + \tau_o\ dACos\theta$$

The negative is taken with pressure force as it is acting in the downward direction while shear force is acting vertically up.

$$\therefore Total\ lift,\ F_L = \int \tau_o dACos\theta - \int pdASin\theta$$

The drag and lift for a body moving in a fluid of density e, at a uniform velocity U are calculated mathematically as

$$F_D = C_D A\frac{\ell U^2}{2}$$

$$F_L = C_L A\frac{\ell U^2}{2}$$

where

C_D= Coefficient of drag

C_L= Coefficient of Lift

A = Area of the body which is the projected area of the body perpendicular to the direction of flow

= largest projected area of the immersed body

Then resultant force on the body, F_R= $\sqrt{F_D^2 + F_L^2}$

Example 1

A flat plate 1.5m x 1.5m moves as 50km/hr in stationary air of density 1.15kg/m^3. If the coefficients of drag and lift are 0.15 and 0.75 respectively. Determine:

(i) The lift force
(ii) The drag force
(iii) The resultant force and
(iv) The power required to keep the plate in motion

Solution

Area of the plate, A = 1.5 x 1.5 = 2.25m^2

Velocity of the plate, U = 50km/hr = $\frac{50 \times 1000}{60 \times 60} = 13.89 m/s$

Density of air, $\ell = 1.15 kg/m^3$

Coefficient of drag, C_D = 0.15

Coefficient of lift, C_L= 0.75

(i) Lift force (F_L) = $C_L A \frac{\ell U^2}{2}$

$$= 0.75 \times 2.25 \times \frac{1.15 \times 13.89^2}{2} = 187.20N$$

(ii) Drag Force (F_D)= $C_D A \frac{\ell U^2}{2}$

$$= 0.15 \times 2.25 \times \frac{1.15 \times 13.89^2}{2} = 37.44N$$

(iii) Resultant force (F_R) = $\sqrt{F_D^2 + F_L^2} = \sqrt{37.44^2 + 187.20^2}$

$$\sqrt{1400 + 35025}$$
$$= 190.85N$$

(iv) Power Required to keep the plate in motion

$$P = \frac{Force\ in\ the\ direction\ of\ motion \times Velocity}{1000} kW$$

$$= \frac{F_D \times U}{1000} = \frac{37.425 \times 13.89}{1000} kW = 0.519kW$$

Example 2

Find the difference in drag force exerted on a flat plate of size 2m x 2m when the plate is moving at a speed of 4m/s normal to its plane in (i) water (ii) air of density 1.24kg/m^3. Coefficient of drag is given as 1.15.

Solution

Area of plate, A = 2 x 2 = 4m^2

Velocity of Plate, U = 4m/s

Coefficient of drag C_D= 1.15

(i) Drag force when the plate is moving in water

$$F_D = C_D \times A \frac{\rho U^2}{2}$$

(ii) Drag force when the plate is moving in air,

$$F_D = C_D \times L \times \frac{\ell U^2}{2}$$

$$= 1.15 \times 4.0 \times 1.24 \times \frac{4^2}{2} = 45.6N ----------(ii)$$

$\therefore$ Difference in drag force = (i) – (ii)

= 36800 – 45.6

= 36754.4N