LECTURE NOTE

MCE 326

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FLUID MECHANICS.

Introduction

Fluid mechanics is that discipline within the broad field of applied mechanics concerned with the behavior of liquids and gases at rest or in motion. This field of mechanics obviously encompasses a vast array of problems that may vary from the study of blood flow in the capillaries 1which are only a few microns in diameter2 to the flow of crude oil across Alaska through an 800-mile-long, 4-ft-diameter pipe. Fluid mechanics principles are needed to explain why airplanes are made streamlined with smooth surfaces for the most efficient flight, whereas golf balls are made with rough surfaces 1dimpled2 to increase their efficiency. Numerous interesting questions can be answered by using relatively simple fluid mechanics ideas. For example:

- How can a rocket generate thrust without having any air to push against in outer space?
- Why can't you hear a supersonic airplane until it has gone past you?
- How can a river flow downstream with a significant velocity even though the slope of the surface is so small that it could not be detected with an ordinary level?
- How can information obtained from model airplanes be used to design the real thing?
- Why does a stream of water from a faucet sometimes appear to have a smooth surface, but sometimes a rough surface?
- How much greater gas mileage can be obtained by improved aerodynamic design of cars and trucks?

The list of applications and questions goes on and on—but you get the point; fluid mechanics is a very important, practical subject. It is very likely that during your career as an engineer you will be involved in the analysis and design of systems that require a good understanding of fluid mechanics. It is hoped that this introductory text will provide a sound foundation of the fundamental aspects of fluid mechanics.

Some Characteristics of Fluids.

One of the first questions we need to explore is, What is a fluid? Or we might ask, What is the difference between a solid and a fluid? We have a general, vague idea of the difference. A solid is "hard"and not easily deformed, whereas a fluid is "soft" and is easily deformed (we can readily move through air). Although quite descriptive, these casual observations of the differences between solids and fluids are not very satisfactory from a scientific or engineering point of view. A closer look at the molecular structure of materials reveals that matter that we commonly think of as a solid (steel, concrete, etc.) has densely spaced molecules with large intermolecular cohesive forces that allow the solid to maintain its shape, and to not be easily deformed. However, for matter that we normally think of as a liquid (water, oil, etc.), the molecules are spaced farther apart, the intermolecular forces are smaller than for solids, and the molecules have more freedom of movement. Thus, liquids can be easily deformed (but not easily compressed) and can be poured into containers or forced through a tube. Gases (air, oxygen, etc.)

have even greater molecular spacing and freedom of motion with negligible cohesive intermolecular forces and as a consequence are easily deformed (and compressed) and will completely fill the volume of any container in which they are placed. Although the differences between solids and fluids can be explained qualitatively on the basis of molecular structure, a more specific distinction is based on how they deform under the action of an external load. Specifically, *a fluid is defined as a substance that deforms continuously when acted on by a shearing stress of any magnitude.* A shearing stress (force per unit area) is created whenever a tangential force acts on a surface. When common solids such as steel or other metals are acted on by a shearing stress, they will initially deform (usually a very small deformation), but they will not continuously deform (flow). However, common fluids such as water, oil, and air satisfy the definition of a fluid—that is, they will flow when acted on by a shearing stress. Some materials, such as slurries, tar, putty, toothpaste, and so on, are not easily classified since they will behave as a solid if the applied shearing stress is small, but if the stress exceeds some critical value, the substance will flow. The study of such materials is called *rheology* and does not fall within the province of classical fluid mechanics. Thus, all the fluids we will be concerned with in this text will conform to the definition of a fluid given previously.

Analysis of Fluid Behavior

The study of fluid mechanics involves the same fundamental laws you have encountered in physics and other mechanics courses. These laws include Newton's laws of motion, conservation of mass, and the first and second laws of thermodynamics. Thus, there are strong similarities between the general approach to fluid mechanics and to rigid-body and deformable body solid mechanics. This is indeed helpful since many of the concepts and techniques of analysis used in fluid mechanics will be ones you have encountered before in other courses. The broad subject of fluid mechanics can be generally subdivided into *fluid statics,* in which the fluid is at rest, and *fluid dynamics,* in which the fluid is moving. In the following chapters we will consider both of these areas in detail. Before we can proceed, however, it will be necessary to define and discuss certain fluid *properties* that are intimately related to fluid behavior. It is obvious that different fluids can have grossly different characteristics. For example, gases are light and compressible, whereas liquids are heavy (by comparison) and relatively incompressible. A syrup flows slowly from a container, but water flows rapidly when poured from the same container. To quantify these differences certain fluid properties are used. In the following several sections the properties that play an important role in the analysis of fluid behavior are considered.

Pressure at a Point

The term pressure is used to indicate the normal force per unit area at a given point acting on a given plane within the fluid mass of interest. A question that immediately arises is how the pressure at a point varies with the orientation of the plane passing through the point. To answer this question, consider the free-body diagram, illustrated in Fig. 1, that was obtained by removing a small triangular wedge of fluid from some arbitrary location within a fluid mass. Since we are considering the situation in which there are no shearing stresses, the only external forces acting on the wedge are due to the pressure and the weight. For simplicity the forces in the x direction are not shown, and the z axis is taken as the vertical axis so the weight acts in the negative z direction. Although we are primarily interested in fluids at rest, to make the analysis as general as possible, we will allow the fluid element to have accelerated motion. The assumption of zero shearing stresses will still be valid so long as the fluid element moves as a rigid body; that is, there is no relative motion between adjacent elements.

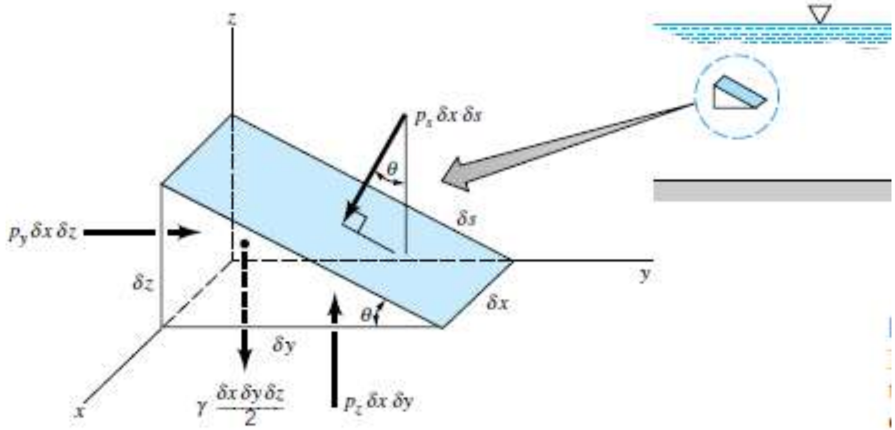

Fig.1.Forces on an arbitrary wedged-shape element of fluid.

The equations of motion (Newton's second law, F = ma) in the *y* and *z* directions are, respectively,

$$\sum F_y = p_y\delta x\delta z - p_s\delta x\delta s\, sin\theta = \rho\frac{\delta x\delta y\delta z}{2}a_y$$

$$\sum F_z = p_z\delta x\delta y - p_s\delta x\delta s\, cos\theta - \gamma\frac{\delta x\delta y\delta z}{2} = \rho\frac{\delta x\delta y\delta z}{2}a_z$$

Where p_s, p_y, and p_z are the average pressures on the faces,γ and ρ are the fluid specific a_y, a_z multiplied by an appropriate area to obtain the force generated by the pressure. It follows from the geometry that

$\delta y = \delta s\, cos\theta,\quad \delta z = \delta s\, sin\theta$, so that the equations of motion can be rewritten as

$$p_y - p_s = \rho a_y\frac{\delta y}{2}$$

$$p_z - p_s = (\rho a_z + \gamma)\frac{\delta z}{2}$$

Since we are really interested in what is happening at a point, we take the limit as δx, δy, and δz approach zero (while maintaining the angle θ), and it follows that $p_y = p_s$, $p_z = p_s$ or $p_s = p_y = p_z$. The angle was arbitrarily chosen so we can conclude that *the pressure at a point in a fluid at rest, or in motion, is independent of direction as long as there are no shearing stresses present*. This important result is known as *Pascal's law* named in honor of Blaise Pascal (1623–1662), a French mathematician who made important contributions in the field of hydrostatics.

Basic Equation for Pressure Field

Although we have answered the question of how the pressure at a point varies with direction, we are now faced with an equally important question—how does the pressure in a fluid in which there are no shearing stresses vary from point to point? To answer this question consider a small rectangular element of fluid removed from some arbitrary position within the mass of fluid of interest as illustrated in Fig. 2. There are two types of forces acting on this element: *surface forces* due to the pressure, and a *body force* equal to the weight of the element. Other possible types of body forces, such as those due to magnetic fields, will not be considered in this text. If we let the pressure at the center of the element be designated as *p*, then the average pressure on the various faces can be expressed in terms of *p* and its derivatives as shown in Fig. 2. We are actually using a Taylor series expansion of the pressure at the element center to approximate the pressures a short distance away and neglecting higher order terms that will vanish as we let δx, δy, and δz approach zero. For simplicity the surface forces in the *x* direction are not shown. The resultant surface force in the *y* direction is

$$\delta F_y = \left(p - \frac{\partial p}{\partial y}\frac{\delta y}{2}\right)\delta x \delta z - \left(p + \frac{\partial p}{\partial y}\frac{\delta y}{2}\right)\delta x \delta z \quad \text{or} \quad \delta F_y = -\frac{\partial p}{\partial y}\ \delta x \delta y\, \delta z$$

Similarly, for the *x* and *z* directions the resultant surface forces are

$$\delta F_x = -\frac{\partial p}{\partial x}\ \delta x \delta y\, \delta z \qquad \delta F_z = -\frac{\partial p}{\partial z}\ \delta x \delta y\, \delta z$$

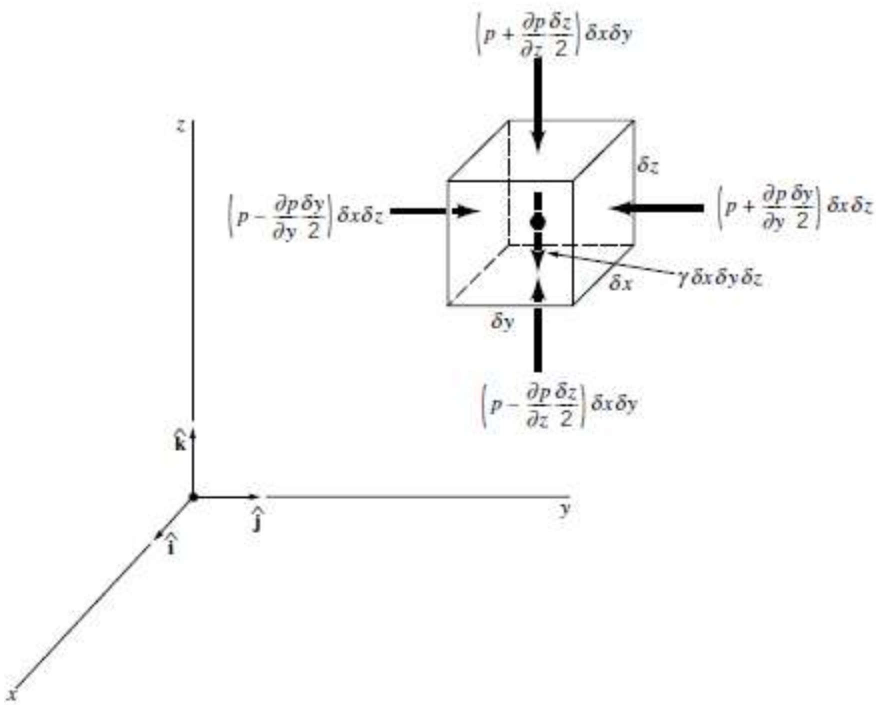

Fig 2. Surface and body forces acting on small fluid element.

The resultant surface force acting on the element can be expressed in vector form as

$$\delta F_s = \delta F_x \hat{\imath} + \delta F_y \hat{\jmath} + \delta F_z \hat{k} \quad \text{or}$$

$$\delta F_s = -\left(\frac{\partial p}{\partial x}\hat{\imath} + \frac{\partial p}{\partial y}\hat{\jmath} + \frac{\partial p}{\partial z}\hat{k}\right)\delta x \delta y \delta z \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots 1$$

Where $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$ are the unit vectors along the coordinate axes shown in Fig. 2. The group of terms in parentheses in Eq. 1 represents in vector form the *pressure gradient* and can be written as

$$\frac{\partial p}{\partial x}\hat{\imath} + \frac{\partial p}{\partial y}\hat{\jmath} + \frac{\partial p}{\partial z}\hat{k} = \nabla p, \text{ Where,}$$

$$\nabla(\,) = \frac{\partial(\,)}{\partial x}\hat{\imath} + \frac{\partial(\,)}{\partial y}\hat{\jmath} + \frac{\partial(\,)}{\partial z}\hat{k}$$

and the symbol ∇ is the *gradient* or "del" vector operator. Thus, the resultant surface force per unit volume can be expressed as

$$\frac{\delta F_s}{\delta x \delta y \delta z} = -\nabla p$$

Since the *z* axis is vertical, the weight of the element is

$$-\delta^{o}W\hat{k} = -\gamma \delta x \delta y \delta z \hat{k}$$

where the negative sign indicates that the force due to the weight is downward (in the negative *z* direction). Newton's second law, applied to the fluid element, can be expressed as

$$\sum \delta F = \delta m\, a$$

Where $\sum \delta F$ represents the resultant force acting on the element, **a** is the acceleration of the element, and δm is the element mass, which can be written as ρδxδyδz. It follows that

$$\sum \delta F = \delta F_s \ -\delta^{o}W\hat{k} = \delta m\, a$$ or,

$$-\nabla p \delta x \delta y \delta z - \gamma \delta x \delta y \delta z \hat{k} = \rho \delta x \delta y \delta z\, a$$

and, therefore,

$$-\nabla p - \gamma \hat{k} = \rho a \qquad \ldots\ldots\ldots\ldots\ldots 2$$

Equation 2. is the general equation of motion for a fluid in which there are no shearing stresses.

Pressure Variation in a Fluid at Rest

For a fluid at rest a = 0 and Eq. 2. reduces to

$\nabla p + \gamma \hat{k} = 0$ or in component form,

$$\frac{\partial p}{\partial x} = 0 \quad \frac{\partial p}{\partial y} = 0 \quad \frac{\partial p}{\partial z} = -\gamma \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ 3$$

These equations show that the pressure does not depend on *x* or *y*. Thus, as we move from point to point in a horizontal plane 1any plane parallel to the *x–y* plane2, the pressure does not change. Since *p* depends only on *z*, the last of Eqs. 3 can be written as the ordinary differential equation

$$\frac{\partial p}{\partial z} = -\gamma \qquad \ldots\ldots\ldots\ldots\ 4$$

Equation 4 is the fundamental equation for fluids at rest and can be used to determine how pressure changes with elevation. This equation indicates that the pressure gradient in the vertical direction is negative; that is, the pressure decreases as we move upward in a fluid at rest. There is no requirement that $-\gamma$ be a constant. Thus, it is valid for fluids with constant specific weight, such as liquids, as well as fluids whose specific weight may vary with elevation, such as air or other gases. However, to proceed with the integration of Eq. 4 it is necessary to stipulate how the specific weight varies with *z*.

Incompressible Fluid

Since the specific weight is equal to the product of fluid density and acceleration of gravity changes in are caused either by a change in or *g*. For most engineering applications the variation in *g* is negligible, so our main concern is with the possible variation in the fluid density. For liquids the variation in density is usually negligible, even over large vertical distances, so that the assumption of constant specific weight when dealing with liquids is a good one. For this instance, Eq. 4 can be directly integrated

$$\int_{p_1}^{p_2} dp = -\gamma \int_{z_1}^{z_2} dz \text{ , to yield,}$$

$$p_2 - p_1 = -\gamma(z_2 - z_1) \text{ or,}$$

$$p_1 - p_2 = \gamma(z_2 - z_1) \quad \text{................ 5}$$

Where $p_1 and\ p_2$ are pressures at the vertical elevations $z_2\ and\ z_1$ as is illustrated in Fig. 3. Equation 5 can be written in the compact form

$$p_1 - p_2 = \gamma h \text{ .... 6}$$

or

$$p_1 = \gamma h + p_2 \quad \text{.................. 7}$$

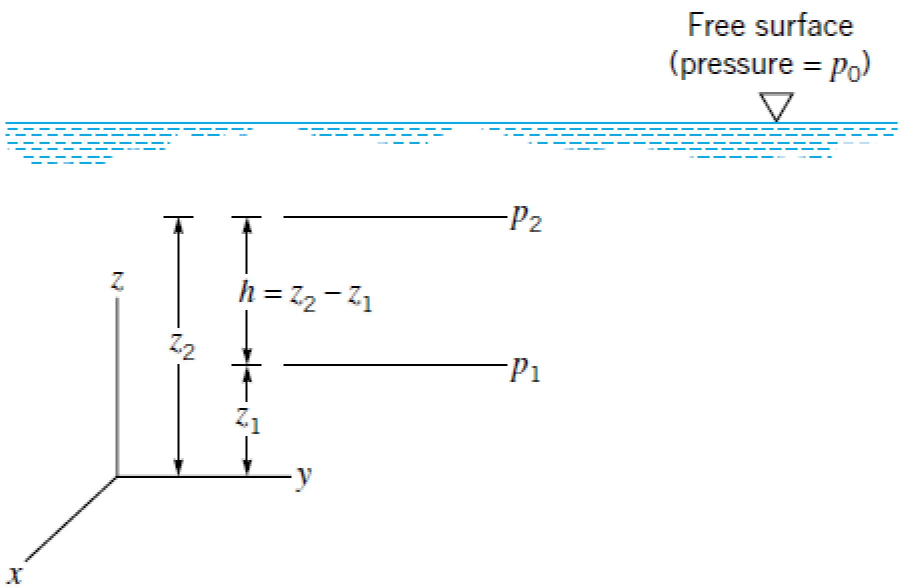

Fig. 3. Notation for pressure variation in a fluid at rest with a free surface.

Where h is the distance, $z_2 - z_1$ which is the depth of fluid measured downward from the location of p_2. This type of pressure distribution is commonly called a *hydrostatic distribution*, and Eq. 7 shows that in an incompressible fluid at rest the pressure varies linearly with depth. The pressure must increase with depth to "hold up" the fluid above it. It can also be observed from Eq. 6 that the pressure difference between two points can be specified by the distance h since

$$h = \frac{p_1 - p_2}{\gamma}$$

In this case h is called the *pressure head* and is interpreted as the height of a column of fluid of specific weight γ required to give a pressure difference $p_1 - p_2$. For example, a pressure difference of 10 psi can be specified in terms of pressure head as 23.1 ft of water(γ = 62.4 Ib/ft^3) or 518 mm of Hg (γ = 133 kN/m^2). When one works with liquids there is often a free surface, as is illustrated in Fig. 3, and it is convenient to use this surface as a reference plane. The reference pressure p_o would correspond to the pressure acting on the free surface (which would frequently be atmospheric pressure), and thus if we let $p_2 = p_0$ in Eq. 7 it follows that the pressure p at any depth h below the free surface is given by the equation:

$$p = \gamma h + p_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 8$$

As is demonstrated by Eq. 7 or 8, the pressure in a homogeneous, incompressible fluid at rest depends on the depth of the fluid relative to some reference plane, and it is *not* influenced by the *size* or *shape* of the tank or container in which the fluid is held. Thus, in Fig. 4 the pressure is the same at all points along the line *AB* even though the container may have the very irregular shape shown in the figure. The actual value of the pressure along *AB* depends only on the depth, *h*, the surface pressure p_o, and the specific weight, γ, of the liquid in the container.

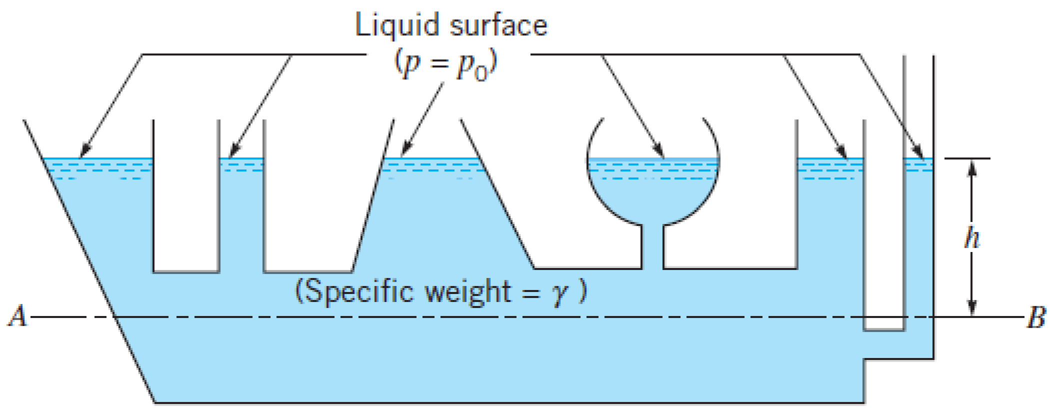

Fig.4. Fluid equilibrium in a container of arbitrary shape.

Operation of hydraulic devices

The required equality of pressures at equal elevations throughout a system is important for the operation of hydraulic jacks, lifts, and presses, as well as hydraulic controls on aircraft and other types of heavy machinery. The fundamental idea behind such devices and systems is demonstrated in Fig. 5. A piston located at one end of a closed system filled with a liquid, such as oil, can be used to change the pressure throughout the system, and thus transmit an applied force F_1 to a second piston where the resulting force F_2 is Since the pressure *p* acting on the faces of both pistons is the same (the effect of elevation changes is usually negligible for this type of hydraulic device), it follows that $F_2 = (A_2/A_1)\ F_1$. The piston area A_2 can be made much larger thanA_1 and therefore a large mechanical advantage can be developed; that is, a small force applied at the smaller piston can be used to develop a large force at the larger piston. The applied force could be created manually through some type of mechanical device, such as a hydraulic jack, or through compressed air acting directly on the surface of the liquid, as is done in hydraulic lifts commonly found in service stations.

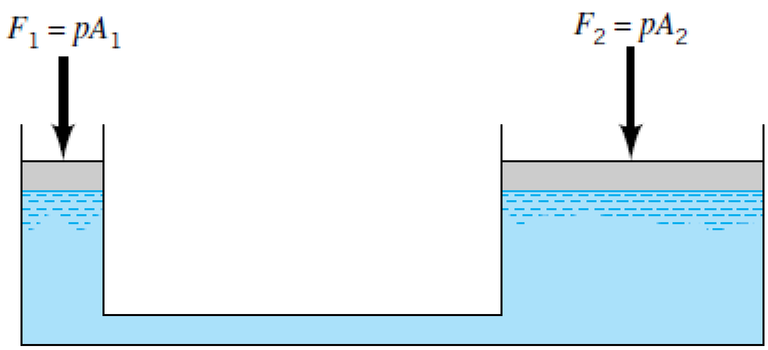

Fig. 5. Transmission of fluid pressure.

Compressible Fluid

We normally think of gases such as air, oxygen, and nitrogen as being compressible fluids since the density of the gas can change significantly with changes in pressure and temperature. Thus, although Eq. 4 applies at a point in a gas, it is necessary to consider the possible variation in γ before the equation can be integrated. However, as was discussed earlier, the specific weights of common gases are small when compared with those of liquids. For example, the specific weight of air at sea level and 60 ^{0}F is 0.0763 Ib/ft^3, (12.014 N/m^3) whereas the specific weight of water under the same conditions is 62.4 Ib/ft^3, Since the specific weights of gases are comparatively small, it follows from Eq. 4 that the pressure gradient in the vertical direction is correspondingly small, and even over distances of several hundred feet the pressure will remain essentially constant for a gas. This means we can neglect the effect of elevation changes on the pressure in gases in tanks, pipes, and so forth in which the distances involved are small. For those situations in which the variations in heights are large, on the order of thousands of feet, attention must be given to the variation in the specific weight. As is described earlier, the equation of state for an ideal (or perfect) gas is

$$P = \rho RT$$

Where *p* is the absolute pressure, *R* is the gas constant, and *T* is the absolute temperature. This relationship can be combined with Eq. 4 to give

$\frac{\partial p}{\partial z} = -\frac{gp}{RT}$, and by separating variables,

$$\int_{P_1}^{p_2} \frac{dp}{p} = ln\frac{p_2}{p_1} = -\frac{g}{R}\int_{z_1}^{z_2} \frac{dz}{T} \quad \ldots\ldots\ldots\ldots\ldots\ldots \; 9$$

Where *g* and *R* are assumed to be constant over the elevation change from z_1 to z_2. Although the acceleration of gravity, *g*, does vary with elevation, the variation is very small and *g* is usually assumed constant at some average value for the range of elevation involved. Before completing the integration, one must specify the nature of the variation of temperature with elevation. For example, if we assume that the temperature has a constant value over the range (*isothermal* conditions), it then follows from Eq. 9 that

$$p_2 = p_1 exp\left[-\frac{g(z_2 - z_1)}{RT_0}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 10$$

This equation provides the desired pressure-elevation relationship for an isothermal layer. For non-isothermal conditions a similar procedure can be followed if the temperature-elevation relationship is known, as is discussed in the following section.

Table 1.

Property	SI Units	BG Units
Temperature, T	288.15 K (15 °C)	518.67 °R (59.00 °F)
Pressure, p	101.33 kPa (abs)	2116.2 lb/ft^2 (abs) [14.696 $lb/in.^2$ (abs)]
Density, ρ	1.225 kg/m^3	0.002377 $slugs/ft^3$
Specific weight, γ	12.014 N/m^3	0.07647 lb/ft^3
Viscosity, μ	1.789×10^{-5} $N \cdot s/m^2$	3.737×10^{-7} $lb \cdot s/ft^2$

[a]Acceleration of gravity at sea level = 9.807 m/s^2 = 32.174 ft/s^2.

Example 1

A mountain lake has an average temperature of and a maximum depth of 40 m. For a barometric pressure of 598 mm Hg, determine the absolute pressure (in pascals) at the deepest part of the lake.

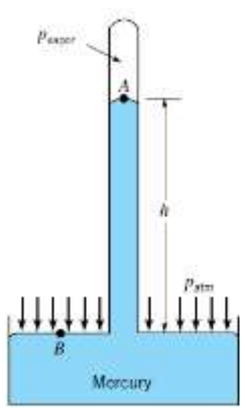

Fig. 6. Mercury barometer

Solution

The pressure in the lake at any depth, h, is given by the equation

$$P = \gamma h + P_0$$

Where P_0 is the pressure at the surface. Since we want the absolute pressure, P_0 will be the local barometric pressure expressed in a consistent system of units; that is

$$\frac{P_{barometric}}{\gamma_{Hg}} = 598\ mm = 0.598\ m$$

and for $\gamma_{Hg} = 133\ kN/m^3$

$P_0 = (0.598\ m)(133\ kN/m^3) = 79.5\ kN/m^2$

$\gamma_{H_2O} = 9.804\ kN/m^3\ at\ 10\ ℃\ and\ therefore$

$P = (9.804\ kN/m^3)(40\ m) + 79.5\ kN/m^2 = 392\ kN/m^2 + 79.5\ kN/m^2 = 472\ kPa$

This simple example illustrates the need for close attention to the units used in the calculation of pressure; that is, be sure to use a *consistent* unit system, and be careful not to add a pressure head (m) to a pressure (Pa).

Pressure Variation in a Fluid with Rigid-Body Motion

Although in this chapter we have been primarily concerned with fluids at rest, the general equation of motion (Eq. 2).

$$-\nabla p - \gamma \hat{k} = \rho a$$

was developed for both fluids at rest and fluids in motion, with the only stipulation being that there were no shearing stresses present. Equation 2. in component form, based on rectangular coordinates with the positive *z* axis being vertically upward, can be expressed as

$$-\frac{\partial p}{\partial x} = \rho a_x \quad -\frac{\partial p}{\partial y} = \rho a_y \quad -\frac{\partial p}{\partial z} = \gamma + \rho a_z \quad \text{………………..} \ 11$$

A general class of problems involving fluid motion in which there are no shearing stresses occurs when a mass of fluid undergoes rigid-body motion. For example, if a container of fluid accelerates along a straight path, the fluid will move as a rigid mass (after the initial sloshing motion has died out) with each particle having the same acceleration. Since there is no deformation, there will be no shearing stresses and, therefore, Eq. 2. applies. Similarly, if a fluid is contained in a tank that rotates about a fixed axis, the fluid will simply rotate with the tank as a rigid body, and again Eq. 2. can be applied to obtain the pressure distribution throughout the moving fluid. Specific results for these two cases (rigid-body uniform motion and rigid-body rotation) are developed in the following two sections. Although problems relating to fluids having rigid-body motion are not, strictly speaking, "fluid statics" problems, they are included in this chapter because, as we will see, the analysis and resulting pressure relationships are similar to those for fluids at rest.

Linear Motion

We first consider an open container of a liquid that is translating along a straight path with a constant acceleration **a** as illustrated in Fig. 7. Since $a_x = 0$ it follows from the first of Eqs. 11 that the pressure gradient in the *x* direction is zero ($\frac{\partial p}{\partial x}$). In the *y* and *z* directions

$$\frac{\partial p}{\partial y} = -\rho a_y \quad \text{…………………..} 1\ 2$$

$$\frac{\partial p}{\partial z} = -\rho(g + a_z) \quad \text{……………………. } 13$$

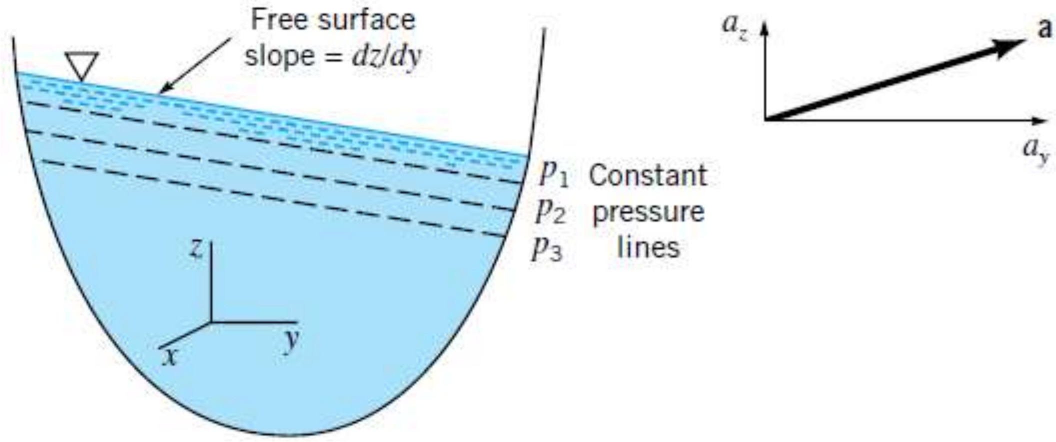

Fig. 7. Linear acceleration of a liquid with a free surface.

The change in pressure between two closely spaced points located at *y*, *z*, and y + dy, z + dz can be expressed as

$dp = \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz$ or in terms of the results from Eqs. 12 and 13

$$dp = -\rho a_y dy - \rho(g + a_z)dz \quad \text{..........14}$$

Along a line of *constant* pressure, $dp = 0$, and therefore from Eq. 14 it follows that the slope of this line is given by the relationship

$$\frac{dz}{dy} = -\frac{a_y}{g + a_z} \quad \text{............15}$$

Along a free surface the pressure is constant, so that for the accelerating mass shown in Fig. 7 the free surface will be inclined if$a_y \neq 0$. In addition, all lines of constant pressure will be parallel to the free surface as illustrated. For the special circumstance in which $a_y = 0, a_z \neq 0$ which corresponds to the mass of fluid accelerating in the vertical direction, Eq. 15 indicates that the fluid surface will be horizontal. However, from Eq. 13 we see that the pressure distribution is not hydrostatic, but is given by the equation

$$\frac{\partial p}{\partial z} = -\rho(g + a_z)$$

For fluids of constant density this equation shows that the pressure will vary linearly with depth, but the variation is due to the combined effects of gravity and the externally induced acceleration, $\rho(g + a_z)$, rather than simply the specific weight ρg. Thus, for example, the pressure along the bottom of a liquid-filled tank which is resting on the floor of an elevator that is accelerating upward will be increased over that which exists when the tank is at rest (or moving with a constant velocity). It is to be noted that for a *freely falling* fluid mass$(a_z = -g)$, the pressure gradients in all three coordinate directions are zero, which means that if the pressure surrounding the mass is zero, the pressure throughout will be zero. The pressure throughout a "blob" of orange juice floating in an orbiting space shuttle (a form of free fall) is zero. The only force holding the liquid together is surface tension.

Example 2

The cross section for the fuel tank of an experimental vehicle is shown in Fig.8. The rectangular tank is vented to the atmosphere, and a pressure transducer is located in its side as illustrated.

During testing of the vehicle, the tank is subjected to a constant linear acceleration, (a) Determine an expression that relates and the pressure at the transducer for a fuel with a (b) What is the maximum acceleration that can occur before the fuel level drops below the transducer?
Solution

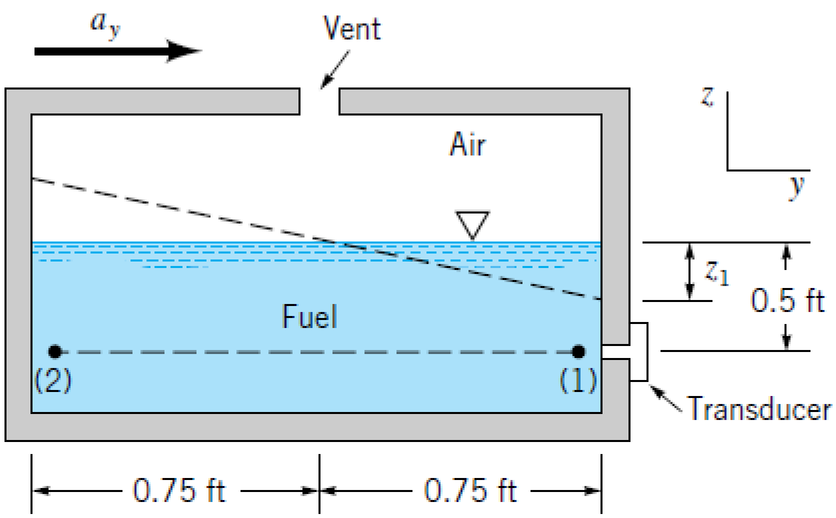

Fig. 8

(a) For a constant horizontal acceleration the fuel will move as a rigid body, and from Eq. 15 the slope of the fuel surface can be expressed as

$$\frac{dz}{dy} = -\frac{a_y}{g}$$

since $a_z = 0$. Thus, for some arbitrary a_y the change in depth, z_1 of liquid on the right side of the tank can be found from the equation

$$-\frac{z_1}{0.75\ ft} = -\frac{a_y}{g}, \text{ or,}$$

$$z_1 = (0.75\ ft)\left(\frac{a_y}{g}\right)$$

Since there is no acceleration in the vertical, *z*, direction, the pressure along the wall varies hydrostatically as shown by Eq. 13. Thus, the pressure at the transducer is given by the relationship

$$P = \gamma h$$

Where *h* is the depth of fuel above the transducer, and therefore

$$P = (0.65)(62.4\ lb/ft^3)[0.5\ ft - (0.75\ ft)(a_y/g)]$$
$$= 20.3 - 30.4\frac{a_y}{g} \text{ ans.}$$

for $z_1 \leq 0.5\ ft$. As written, *p* would be given in lb/ft^2.

(b) The limiting value for (when the fuel level reaches the transducer) can be found from the equation

$0.5\ ft = (0.75\ ft)\left[\frac{(a_y)_{max}}{g}\right]$ or,

$(a_y)_{max} = \frac{2g}{3}$

and for standard acceleration of gravity

$(a_y)_{max} = \frac{2}{3}(32.2\ ft/s^2) = 21.5\ ft/s^2$ Ans.

Note that the pressure in horizontal layers is not constant in this example since $\partial P/\partial y = -\rho a_y \neq 0$. Thus, for example, $P_1 \neq P_2$.

Rigid-Body Rotation

After an initial "start-up" transient, a fluid contained in a tank that rotates with a constant angular velocity ω about an axis as is shown in Fig. 9 will rotate with the tank as a rigid body. It is known from elementary particle dynamics that the acceleration of a fluid particle located at a distance *r* from the axis of rotation is equal in magnitude to $r\omega^2$ and the direction of the acceleration is toward the axis of rotation as is illustrated in the figure. Since the paths of the fluid particles are circular, it is convenient to use cylindrical polar coordinates *r*, θ and *z*, defined in the insert in Fig. 9. From the lecture, it will be observed that in terms of cylindrical coordinates the pressure gradient ∇p can be expressed as

$$\nabla p = \frac{\partial P}{\partial r}\widehat{e_r} + \frac{1}{r}\frac{\partial P}{\partial \theta}\hat{e}_\theta + \frac{\partial P}{\partial z}\hat{e}_z \ \ldots\ldots\ldots\ldots 16$$

Thus, in terms of this coordinate system

$$a_r = -r\omega^2\hat{e}_r \quad a_\theta = 0 \quad a_z = 0$$

and from Eq. 2.

$$\frac{\partial P}{\partial r} = \rho r\omega^2 \quad \frac{\partial P}{\partial \theta} = 0 \quad \frac{\partial P}{\partial r} = -\gamma \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots 17$$

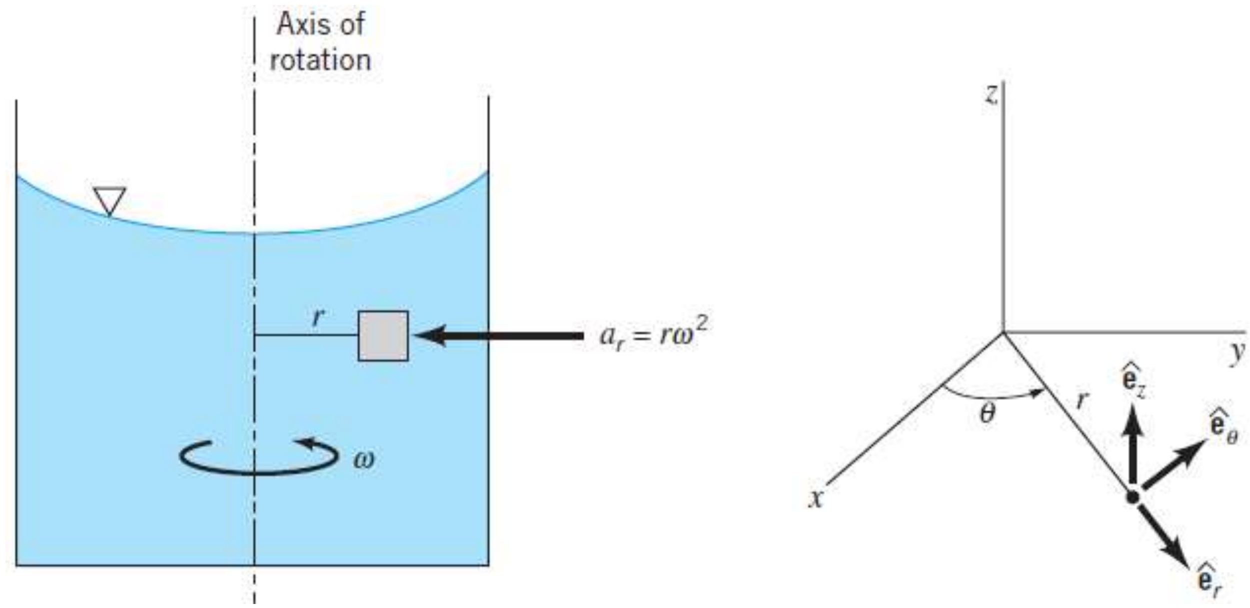

Fig. 9. Rigid-body rotation of a liquid in a tank.

These results show that for this type of rigid-body rotation, the pressure is a function of two variables *r* and *z*, and therefore the differential pressure is

$$dp = \frac{\partial P}{\partial r}dr + \frac{\partial P}{\partial z}dz \qquad \text{or}$$

$$dp = \rho r\omega^2 dr - \gamma dz \quad \ldots\ldots\ldots\ldots 18$$

Along a surface of constant pressure, such as the free surface,dp = 0 so that from Eq. 18
(using $\gamma = \rho g$)

$$\frac{dz}{dr} = \frac{r\omega^2}{g}$$

and, therefore, the equation for surfaces of constant pressure is

$$z = \frac{\omega^2 r^2}{2g} + constant \quad \ldots\ldots\ldots\ldots\ldots\ldots 19$$

This equation reveals that these surfaces of constant pressure are parabolic as illustrated in
Fig. 10. Integration of Eq. 18 yields
$\int dp = \rho\omega^2 \int rdr - \gamma \int dz$, or,

$$P = \frac{\rho\omega^2 r^2}{2} - \gamma z + constant \quad \ldots\ldots\ldots\ldots\ldots 20$$

Where the constant of integration can be expressed in terms of a specified pressure at some arbitrary point r_0, z_0. This result shows that the pressure varies with the distance from the axis of rotation, but at a fixed radius, the pressure varies hydrostatically in the vertical direction as shown in Fig. 10.

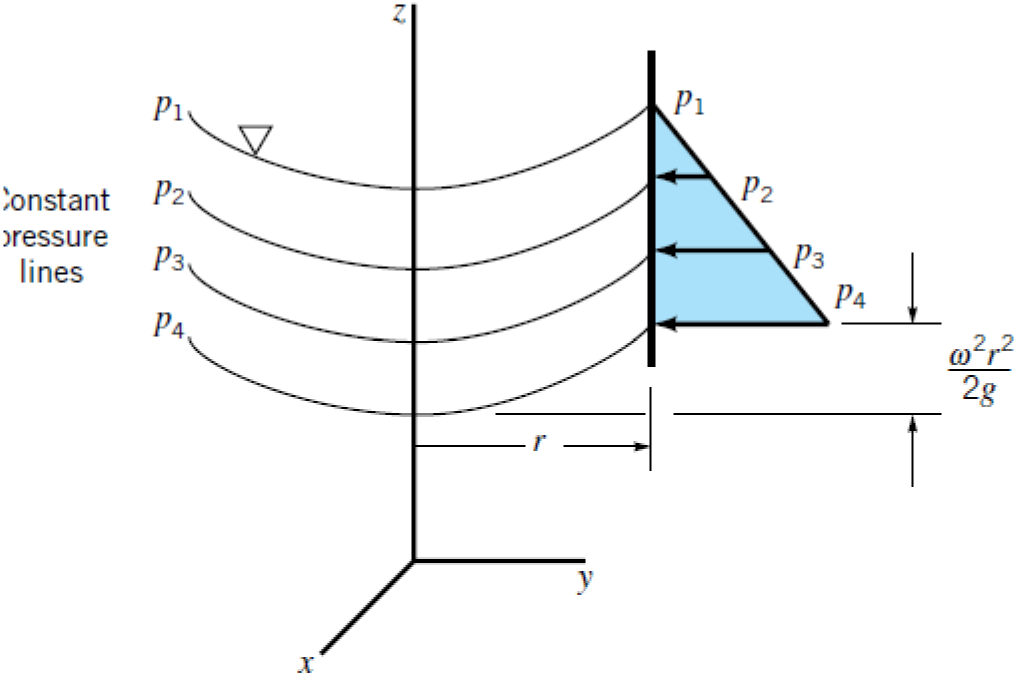

Fig. 10. Pressure distribution in rotating liquid

Example 3

It has been suggested that the angular velocity, ω, of a rotating body or shaft can be measured by attaching an open cylinder of liquid, and measuring with some type of depth gage the change in the fluid level, $H - h_0$, caused by the rotation of the fluid. Determine the relationship between this change in fluid level and the angular velocity.

Solution

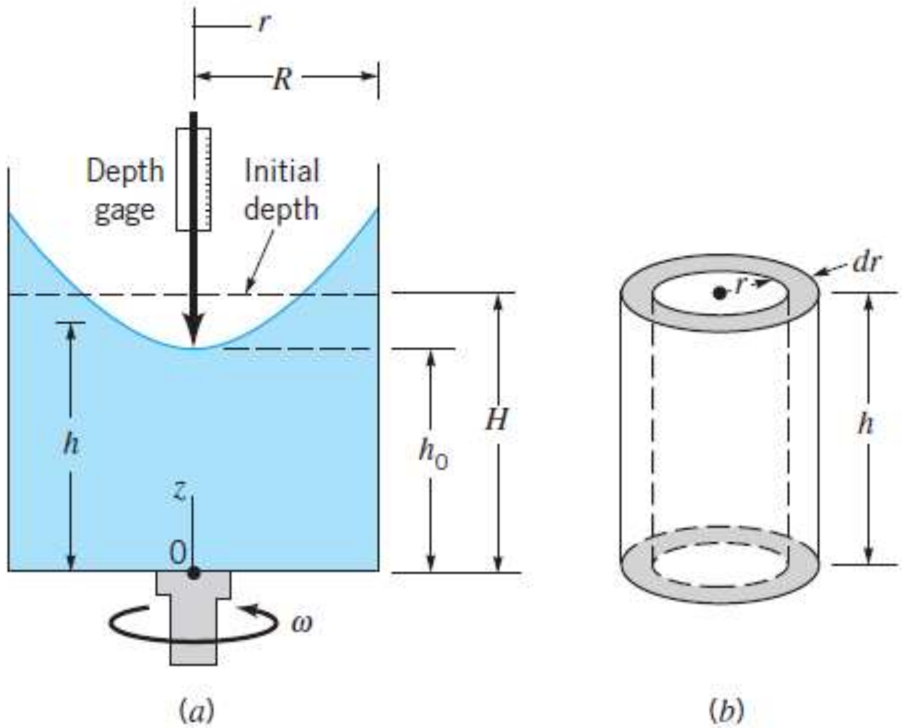

Fig.11, Ex. 3

The height, *h*, of the free surface above the tank bottom can be determined from Eq. 19, and it follows that

$$h = \frac{\omega^2 r^2}{2g} + h_0$$

The initial volume of fluid in the tank, $\overline{\nabla}_i$, is equal to

$$\overline{\nabla}_i = \pi R^2 H$$

The volume of the fluid with the rotating tank can be found with the aid of the differential element shown in Fig.11 Ex. 3. This cylindrical shell is taken at some arbitrary radius, *r*, and its volume is

$$d\,\overline{\nabla} = 2\pi r h\, dr$$

The total volume is, therefore

$$\overline{\nabla} = 2\pi \int_0^R r\left(\frac{\omega^2 r^2}{2g} + h_0\right) dr = \frac{\pi\omega^2 R^4}{4g} + \pi R^2 h_0$$

Since the volume of the fluid in the tank must remain constant (assuming that none spills over the top), it follows that

$$\pi R^2 H = \frac{\pi\omega^2 R^4}{4g} + \pi R^2 h_0 \quad \text{or,}$$

$$H - h_0 = \frac{\omega^2 R^2}{4g} \quad \text{Ans.}$$

This is the relationship we were looking for. It shows that the change in depth could indeed be used to determine the rotational speed, although the relationship between the change in depth and speed is not a linear one.