2.15. Frequency of Under Damped Forced Vibrations

Consider a system consisting of spring, mass and damper as shown in Fig. 22.

Let the system is acted upon by an external periodic (*i.e.* simple harmonic) disturbing force,

$$F_x = F\cos\omega.t$$

where F = Static force, and

ω = Angular velocity of the periodic disturbing force.

When the system is constrained to move in vertical guides, it has only one degree of freedom. Let at sometime

t, the mass is displaced downwards through a distance x from its mean position. Using the symbols as discussed in the previous article, the equation of motion may be written as

$$m\times\frac{d^2x}{dt^2} = -c\times\frac{dx}{dt} - s.x + F\cos\omega t$$

or $$m\times\frac{d^2x}{dt^2} + c\times\frac{d_x}{d_t} + s.x = F\cos\omega t$$

This equation of motion may be solved either by differential equation method or by graphical method as discussed below:

1. *Differential equation method*

The equation ***(i)*** is a differential equation of the second degree whose right hand side is some function in t. The solution of such type of differential equation consists of two parts; one part is the complementary function and the second is particular integral. Therefore the solution may be written as

$$x = x_1 + x_2$$

where x_1 = Complementary function, and

x_2 = Particular integral.

The complementary function is same as discussed in the previous article, *i.e.*

$$x_1 = ce^{-at}\cos(\omega_d t\theta)$$

$$x_2 = B_1 \sin\omega.t + B_2 \cos\omega.t$$

where C and θ are constants. Let us now find the value of particular integral as discussed below:

Let the particular integral of equation ***(i)*** is given by

$$x_2 = B_1\sin\omega.t + B_2\cos.t \ldots \text{(where } B_1 \text{ and } B_2 \text{ are constants)}$$

$$\therefore \qquad \frac{dx}{dt} = B_1.\omega\cos\omega.t - B_2.\omega\sin\omega.t$$

and
$$\frac{d^2x}{dt^2} = -B_1.\omega^2\sin\omega.t - B_2.\omega^2\cos\omega.t$$

Substituting these values in the given differential equation (*i*), we get

$$m(-B_1.\omega^2\sin\omega.t - B_2.\omega^2\cos\omega.t) + c(B_1.\omega\cos\omega.t - B_2.\omega\sin\omega.t) + s(B_1\sin\omega.t + B_2\cos\omega.t)$$
$$= F\cos\omega.t$$

or
$$(-m.B_1.\omega^2 - c.\omega.B_2 + s.B_1)\sin\omega.t + (-m.\omega^2.B_2 + c.\omega.B_1 + s.B_2)\cos\omega.t$$
$$= F\cos\omega.t$$

or
$$\left[(s - m.\omega^2)B_1 - c.\omega.B_2\right]\sin\omega.t + \left[c.\omega.B_1 + (s - m.\omega^2)B_2\right]\cos\omega.t$$
$$= F\cos\omega.t + 0\sin\omega.t$$

Comparing the coefficients of sin ωt and cos ωt on the left hand side and right hand side separately, we get

$$(s - m.\omega^2)B_1 - c.\omega.B_2 = 0 \qquad \ldots (iii)$$

and
$$c.\omega.B_1 + (s - m.\omega^2)B_2 = F \qquad \ldots (iv)$$

Now from equation (*iii*)

$$(s - m.\omega^2)B_1 = c.\omega.B_2$$

$$\therefore \qquad B_2 = \frac{S - m.\omega^2}{c.\omega} \times B_1 \qquad \ldots\ldots(V)$$

Substituting the value of B_2 in equation iv

$$C.\ \omega\ .B_1 + \frac{(S-m-\omega^2\ )\ (s-m-\omega^2}{s.\omega} \times B_1 = F_1$$

$$c^2.\omega^2.B_1 + (s - m.\omega^2)^2 B_1 = c.\omega.F$$

$$B_1\left[c^2.\omega^2 + (s - m.\omega^2)^2\right] = c.\omega.F$$

$$\therefore \qquad B_1 = \frac{c.\omega.F}{c^2.\omega^2 + (s - m.\omega^2)^2}$$

and
$$B_2 = \frac{s - m\omega^2}{c.\omega} \times \frac{c.\omega.F}{c^2.\omega^2 + (s - m.\omega^2)^2} \qquad \ldots \text{[From equation } (v)\text{]}$$

$$= \frac{F(s - m.\omega^2)}{c^2.\omega^2 + (s - m.\omega^2)^2}$$

$\therefore$ The particular integral of the differential equation (*i*) is

$$x_2 = B_1 \sin \omega.t + B_2 \cos \omega.t$$

$$= \frac{c.\omega.F}{c^2.\omega^2 + (s - m.\omega^2)^2} \times \sin \omega.t + \frac{F(s - m.\omega^2)}{c^2.\omega^2 + (s - m.\omega^2)^2} \times \cos \omega.t$$

$$= \frac{F}{c^2.\omega^2 + (s - m.\omega^2)^2} \left[c.\omega \sin \omega.t + (s - m.\omega^2) \cos \omega.t \right] \quad \ldots (vi)$$

Let $c.\omega = X \sin \phi;$ and $s - m.\omega^2 = X \cos \phi$

$\therefore$ $X = \sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}$ . . . (By squaring and adding)

and $\tan \phi = \dfrac{c.\omega}{s - m.\omega^2}$ or $\phi = \tan^{-1}\left(\dfrac{c.\omega}{s - m.\omega^2}\right)$

Now the equation (*vi*) may be written as

$$x_2 = \frac{F}{c^2.\omega^2 + (s - m.\omega^2)^2} \left[X \sin \phi . \sin \omega t + X \cos \phi \cos \omega t \right]$$

$$= \frac{F.X}{c^2.\omega^2 + (s - m.\omega^2)^2} \times \cos(\omega.t - \phi)$$

$$= \frac{F\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}}{c^2.\omega^2 + (s - m.\omega^2)^2} \times \cos(\omega.t - \phi)$$

$$= \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}} \times \cos(\omega.t - \phi)$$

$\therefore$ The complete solution of the differential equation (*i*) becomes

$$x = x_1 + x_2$$

$$= C.e^{-at} \cos(\omega_d.t - \theta) + \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}} \times \cos(\omega.t - \phi)$$

In actual practice, the value of the complementary function $x1$ at any time t is much smaller as compared to particular integral x_2. Therefore, the displacement x, at any time t, is given by the particular integral x_2 only.

$$\therefore \qquad x = \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}} \times \cos(\omega.t - \phi) \qquad \text{... (vii)}$$

This equation shows that motion is simple harmonic whose circular frequency is ω and the amplitude is $\dfrac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}}$.

A little consideration will show that the frequency of forced vibration is equal to the angular velocity of the periodic force and the amplitude of the forced vibration is equal to the maximum displacement of vibration.

$\therefore$ Maximum displacement or the amplitude of forced vibration,

$$x_{max} = \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}} \qquad \text{. . . (viii)}$$

Notes : 1. The equations *(vii)* and *(viii)* hold good when steady vibrations of constant amplitude takes place.

2. The equation *(viii)* may be written as

$$x_{max} = \frac{F/s}{\sqrt{\dfrac{c^2.\omega^2}{s^2} + \dfrac{(s - m.\omega^2)^2}{s^2}}}$$

. . . (Dividing the numerator and denominator by s)

$$= \frac{x_o}{\sqrt{\frac{c^2.\omega^2}{s^2} + \left(1 - \frac{m.\omega^2}{s}\right)^2}} \qquad \ldots \text{(Substituting } F/s = x_o\text{)}$$

where x_0 is the deflection of the system under the static force F. We know that the natural frequency of free vibrations is given by

$$(\omega_n)^2 = s/m$$

$$\therefore \qquad x_{max} = \frac{x_o}{\sqrt{\frac{c^2.\omega^2}{s^2} + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}} \qquad \ldots (ix)$$

3. When damping is negligible, then $c = 0$.

$$\therefore \qquad x_{max} = \frac{x_o}{1 - \frac{\omega^2}{(\omega_n)^2}} = \frac{x_o(\omega_n)^2}{(\omega_n)^2 - \omega^2} = \frac{x_o \times s/m}{(\omega_n)^2 - \omega^2} \qquad \ldots \left[\because (\omega_n)^2 = s/m\right]$$

$$\therefore \qquad = \frac{F}{m\left[(\omega_n)^2 - \omega^2\right]} \qquad \ldots (\because F = x_o.s) \ \ldots (x)$$

4. At resonance $\omega = \omega_n$. Therefore the angular speed at which the resonance occurs is $\omega = \omega_n = \sqrt{\frac{s}{n}}$ rad/s

And $X_{max} = X_0 \times \frac{s}{c\omega_n}$ …[from eqt. ix]

2.16. Magnification Factor or Dynamic Magnifier

It is the ratio of ***maximum displacement of the forced vibration (x_{max}) to the deflection due to the static force F(xo).*** We have proved in the previous article that the maximum displacement or the amplitude of forced vibration,

$$x_{max} = \frac{x_o}{\sqrt{\frac{c^2.\omega^2}{s^2} + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}}$$

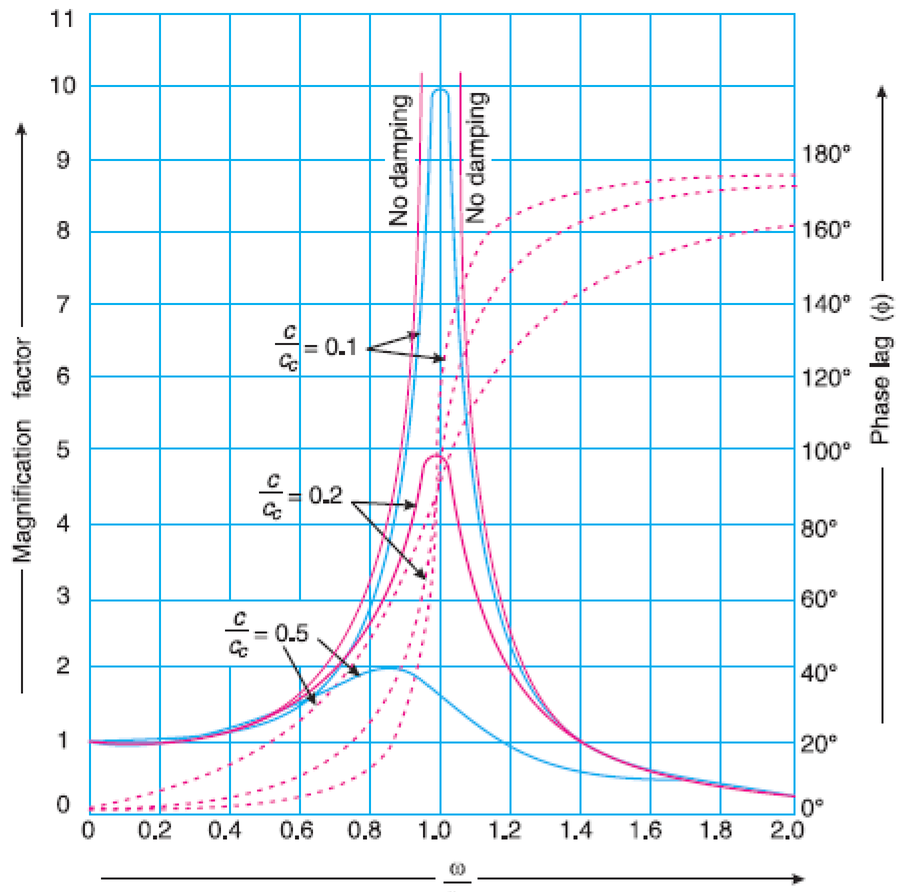


Fig.22 Relationship between magnification factor and phase angle for different values of ω/ω_n.

$\therefore$ Magnification factor or dynamic magnifier,

$$D = \frac{x_{max}}{x_o} = \frac{1}{\sqrt{\frac{c^2.\omega^2}{s^2} + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}} \quad \ldots (i)$$

$$= \frac{1}{\sqrt{\left(\frac{2c.\omega}{c_c.\omega_n}\right)^2 + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}}$$

$$\ldots \left[\because \frac{c.\omega}{s} = \frac{2c.\omega}{2m \times \frac{s}{m}} = \frac{2c.\omega}{2m(\omega_n)^2} = \frac{2c.\omega}{c_c.\omega_n}\right]$$

The magnification factor or dynamic magnifier gives the factor by which the static deflection produced by a force F (*i.e.* x_o) must be multiplied in order to obtain the maximum amplitude of the forced vibration (*i.e.* x_{max}) by the harmonic force $F \cos \omega.t$

$$\therefore \quad x_{max} = x_o \times D$$

Fig. 22 shows the relationship between the magnification factor (D) and phase angle θ for different value of ω/ω_n and for values of damping factor c/c_c = 0.1, 0.2 and 0.5.

Notes: 1. If there is no damping (*i.e.* if the vibration is undamped), then c = 0. In that case, magnification factor,

$$D = \frac{x_{max}}{x_o} = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}} = \frac{(\omega_n)^2}{(\omega_n)^2 - \omega^2}$$

2. At resonance, $\omega = \omega_n$. Therefore magnification factor,

$$D = \frac{x_{max}}{x_o} = \frac{s}{c.\omega_n}$$

Example 3.0. *A single cylinder vertical petrol engine of total mass 300 kg is mounted upon a steel chassis frame and causes a vertical static deflection of 2 mm. The reciprocating parts of the engine has a mass of 20 kg and move through a vertical stroke of 150 mm with simple harmonic motion. A dashpot is provided whose damping resistance is directly proportional to the and amounts to 1.5 kN per metre per second. Considering that the steady state of vibration is reached; determine:* **1.** *the amplitude of forced vibrations, when the driving shaft of the* *engine rotates at 480 r.p.m., and* **2**. *the speed of the driving shaft at which resonance will occur.*

Solution: Given. m = 300 kg; δ = 2 mm = 2 × 10–3 m; m_1 = 20 kg; l = 150 mm = 0.15 m; c = 1.5 kN /m/s =1500 N/m/s; N = 480 r.p.m. or $\omega = 2\pi \times 480 / 60 = 50.3$ rad/s

1. *Amplitude of the forced vibrations*

We know that stiffness of the frame,

$$s = m.g / \delta = 300 \times 9.81/2 \times 10^{-3} = 1.47 \times 10^6 \text{ N/m}$$

Since the length of stroke (l) = 150 mm = 0.15 m, therefore radius of crank,

$$r = l / 2 = 0.15 / 2 = 0.075 \text{ m}$$

We know that the centrifugal force due to the reciprocating parts or the static force,

$$F = m_1.\omega^2.r = 20\ (50.3)^2\ 0.075 = 3795 \text{ N}$$

∴ Amplitude of the forced vibration (maximum),

$$x_{max} = \frac{F}{\sqrt{c^2.\omega^2 + (s - m\omega^2)^2}}$$

$$= \frac{3795}{\sqrt{(1500)^2 (50.3)^2 + [1.47 \times 10^6 - 300\,(50.3)^2]^2}}$$

$$= \frac{3795}{\sqrt{5.7 \times 10^9 + 500 \times 10^9}} = \frac{3795}{710 \times 10^3} = 5.3 \times 10^{-3} \text{ m}$$

= 5.3 mm **Ans.**

2. *Speed of the driving shaft at which the resonance occurs*

Let N = Speed of the driving shaft at which the resonance occurs in r.p.m.

We know that the angular speed at which the resonance occurs,

$$\omega = \omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{1.47 \times 10^6}{300}} = 70 \text{ rad/s}$$

$$\therefore \quad N = \omega \times 60 / 2\pi = 70 \times 60 / 2\pi = 668.4 \text{ r.p.m.}$$ **Ans.**

Example 3.1. *A mass of 10 kg is suspended from one end of a helical spring, the other end being fixed. The stiffness of the spring is 10 N/mm. The viscous damping causes the amplitude to decrease to one-tenth of the initial value in four complete oscillations. If a periodic force of 150 cos 50 t N is applied at the mass in the* *vertical direction, find the amplitude of the forced vibrations. What is its value of resonance?*

Solution. Given : $m = 10$ kg ; $s = 10$ N/mm $= 10 \times 10^3$ N/m ; $x_5 = \dfrac{x_1}{10}$

Since the periodic force, $F_x = F\cos\omega t = 150\cos 50t$, therefore

Static force, $F = 150$ N

and angular velocity of the periodic disturbing force,

$$\omega = 50 \text{ rad/s}$$

We know that angular speed or natural circular frequency of free vibrations

$$\omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{10\times10^3}{10}} = 31.6 \text{ rad/s}$$

Amplitude of the forced vibrations

Since the amplitude decreases to 1/10th of the initial value in four complete oscillations, therefore, the ratio of initial amplitude (x_1) to the final amplitude after four complete oscillations (x_5) is given by

$$\frac{x_1}{x_5} = \frac{x_1}{x_2}\times\frac{x_2}{x_3}\times\frac{x_3}{x_4}\times\frac{x_4}{x_5} = \left(\frac{x_1}{x_2}\right)^4 \qquad \ldots \left(\because \frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \frac{x_4}{x_5}\right)$$

$$\therefore \qquad \frac{x_1}{x_2} = \left(\frac{x_1}{x_5}\right)^{1/4} = \left(\frac{x_1}{x_1/10}\right)^{1/4} = (10)^{1/4} = 1.78 \qquad \ldots \left(x_5 = \frac{x_1}{10}\right)$$

We know that

$$\log_e\left(\frac{x_1}{x_2}\right) = a\times\frac{2\pi}{\sqrt{(\omega_n)^2 - a^2}}$$

$$\log_e 1.78 = a\times\frac{2\pi}{\sqrt{(31.6)^2 - a^2}} \quad \text{or} \quad 0.576 = \frac{a\times2\pi}{\sqrt{1000 - a^2}}$$

Squaring both sides and rearranging,

$$39.832\, a^2 = 332 \quad \text{or} \quad a^2 = 8.335 \quad \text{or} \quad a = 2.887$$

We know that $a = c/2m$ or $c = a \times 2m = 2.887 \times 2 \times 10 = 57.74$ N/m/s

and deflection of the system produced by the static force F,

$$x_o = F/s = 150/10 \times 10^3 = 0.015 \text{ m}$$

We know that amplitude of the forced vibrations.

$$x_{max} = \frac{x_o}{\sqrt{\frac{c^2.\omega^2}{s^2} + \left[1 - \frac{\omega^2}{(\omega_n)^2}\right]^2}}$$

$$= \frac{0.015}{\sqrt{\frac{(57.74)^2(50)^2}{(10\times10^3)^2} + \left[1 - \left(\frac{50}{31.6}\right)^2\right]^2}} = \frac{0.015}{\sqrt{0.083 + 2.25}}$$

$$= \frac{0.015}{1.53} = 9.8\times10^{-3} \text{ m} = 9.8 \text{ mm } \textbf{Ans.}$$

Amplitude of forced vibrations at resonance

We know that amplitude of forced vibrations at resonance,

$$x_{max} = x_0 \times \frac{s}{c.\omega_n} = 0.015\times\frac{10\times10^3}{57.54\times31.6} = 0.0822 \text{ m} = 82.2 \text{ mm } \textbf{Ans.}$$

Example 3.2. *A body of mass 20 kg is suspended from a spring which deflects 15 mm under this load. Calculate the frequency of free vibrations and verify that a viscous damping force amounting to approximately 1000 N at a speed of 1 m/s is just-sufficient to make the motion aperiodic. If when damped to this extent, the body is subjected to a disturbing force with a maximum value of 125 N making 8 cycles/s, find the amplitude of the ultimate motion.*

Solution. Given: m = 20 kg; ∂ = 15 mm = 0.015 m ; c = 1000 N/m/s ; F = 125 N; f = 8 cycles/s

Frequency of free vibrations

We know that frequency of free vibrations,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} = \frac{1}{2\pi}\sqrt{\frac{9.81}{0.015}} = 4.07 \text{ Hz} \quad \textbf{Ans.}$$

The critical damping to make the motion aperiodic is such that damped frequency is zero, *i.e.*

$$\left(\frac{c}{2m}\right)^2 = \frac{s}{m}$$

$$\therefore \quad c = \sqrt{\frac{s}{m}\times4m^2} = \sqrt{4\,s.m} = \sqrt{4\times\frac{m.g}{\delta}\times m} \qquad \ldots \left(\because\ s = \frac{m.g}{\delta}\right)$$

$$= \sqrt{4\times\frac{20\times9.81}{0.015}\times20} = 1023 \text{ N/m/s}$$

This means that the viscous damping force is 1023 N at a speed of 1 m/s. Therefore a viscous damping force amounting to approximately 1000 N at a speed of 1 m/s is just sufficient to make the motion aperiodic. **Ans.**

Amplitude of ultimate motion

We know that angular speed of forced vibration,

$$\omega = 2\pi \times f = 2\pi \times 8 = 50.3 \text{ rad/s}$$

and stiffness of the spring, $s = m.g/\delta = 20 \times 9.81 / 0.015 = 13.1 \times 10^3$ N/m

$\therefore$ Amplitude of ultimate motion *i.e.* maximum amplitude of forced vibration,

$$x_{max} = \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}}$$

$$= \frac{125}{\sqrt{(1023)^2 (50.3)^2 + [13.1 \times 10^3 - 20 (50.3)^2]^2}}$$

$$= \frac{125}{\sqrt{2600 \times 10^6 + 1406 \times 10^6}} = \frac{125}{63.7 \times 10^3} = 1.96 \times 10^{-3} \text{ m}$$

$= 1.96$ mm **Ans.**

Example 3.3 *A machine part of mass 2 kg vibrates in a viscous medium. Determine the damping coefficient when a harmonic exciting force of 25 N results in a resonant amplitude of*
12.5 mm with a period of 0.2 second. If the system is excited by a harmonic force of frequency
4 Hz what will be the percentage increase in the amplitude of vibration when damper is removed as compared with that with damping.

Solution. Given: $m = 2$ kg; $F = 25$ N; Resonant $x_{max} = 12.5$ mm $= 0.0125$ m;

$t_p = 0.2$ s ; $f = 4$ Hz

Damping coefficient

Let c = Damping coefficient in N/m/s.

We know that natural circular frequency of the exciting force,

$$\omega_n = 2\pi/t_p = 2\pi / 0.2 = 31.42 \text{ rad/s}$$

We also know that the maximum amplitude of vibration at resonance (x_{max}),

$$0.0125 = \frac{F}{C.\omega_N} = \frac{25}{C \times 31.42} = \frac{0.796}{C} \quad or\ c = 63.7$$

Percentage increase in amplitude

Since the system is excited by a harmonic force of frequency (f) = 4 Hz, therefore corresponding circular frequency

$$\omega = 2\pi \times f = 2\pi \times 4 = 25.14 \text{ rad/s}$$

We know that maximum amplitude of vibration with damping,

$$x_{max} = \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}}$$

$$= \frac{25}{\sqrt{(63.7)^2 (25.14)^2 + [2\,(31.42)^2 - 2\,(25.14)^2]^2}}$$

$$\ldots \left[\because \ (\omega_n)^2 = s/m \quad \text{or} \quad s = m(\omega_n)^2\right]$$

$$= \frac{25}{\sqrt{2.56 \times 10^6 + 0.5 \times 10^6}} = \frac{25}{1749} = 0.0143 \text{ m} = 14.3 \text{ mm}$$

and the maximum amplitude of vibration when damper is removed,

$$x_{max} = \frac{F}{m\left[(\omega_n)^2 - \omega^2\right]} = \frac{25}{2[(31.42)^2 - (25.14)^2]} = \frac{25}{710} = 0.0352 \text{ m}$$

$$= 35.2 \text{ mm}$$

$\therefore$ Percentage increase in amplitude

$$= \frac{35.2 - 14.3}{14.3} = 1.46 \quad \text{or} \quad 146\% \ \textbf{Ans.}$$

2.17. Vibration Isolation and Transmissibility

A little consideration will show that when an unbalanced machine is installed on the foundation, it produces vibration in the foundation. In order to prevent these vibrations or to minimize the transmission of forces to the foundation, the machines are mounted on springs and dampers or on some vibration isolating material, as shown in Fig.23. The arrangement is assumed to have one degree of freedom, *i.e.* it can move up and down only. It may be noted that when a periodic (*i.e.* simple harmonic) disturbing force $F \cos \omega.t$ is applied to a machine of mass m supported by a spring of stiffness s, then the force is transmitted by means of the spring and the damper or dashpot to the fixed support or foundation. The ratio of the force transmitted (FT) to the force applied (F) is known as the ***isolation factor*** or ***transmissibility ratio*** of the spring support.

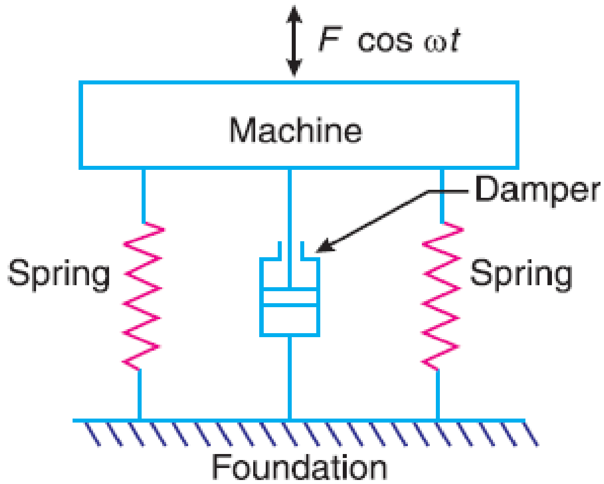


Fig. 23. Vibration isolation.

We have discussed above that the force transmitted to the foundation consists of the following two forces:

1. Spring force or elastic force which is equal to $s.\ x_{max}$, and

2. Damping force which is equal to $c.\omega\ x_{max}$.

Since these two forces are perpendicular to one another, as shown in Fig23., therefore the force transmitted,

$$F_T = \sqrt{(s.x_{max})^2 + (c.\omega.x_{max})^2}$$

$$= x_{max}\sqrt{s^2 + c^2.\omega^2}$$

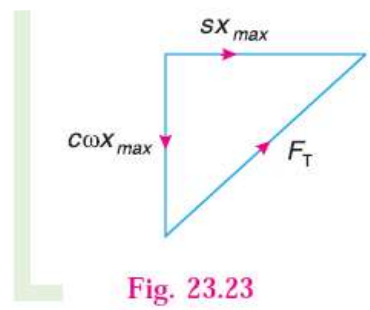


Fig. 23.23

$\therefore$ Transmissibility ratio,

$$\varepsilon = \frac{F_T}{F} = \frac{x_{max}\sqrt{s^2 + c^2.\omega^2}}{F}$$

We know that

$$x_{max} = x_o \times D = \frac{F}{s} \times D \qquad \ldots \left(\because\ x_o = \frac{F}{s}\right)$$

$$\therefore \qquad \varepsilon = \frac{D}{s}\sqrt{s^2 + c^2.\omega^2} = D\sqrt{1 + \frac{c^2.\omega^2}{s^2}}$$

$$= D\sqrt{1 + \left(\frac{2c}{c_c} \times \frac{\omega}{\omega_n}\right)^2} \qquad \ldots \left(\because\ \frac{c.\omega}{s} = \frac{2c}{c_c} \times \frac{\omega}{\omega_n}\right)$$

We have seen in Art. 2.18 that the magnification factor,

$$D = \frac{1}{\sqrt{\left(\frac{2c.\omega}{c_c.\omega_n}\right)^2 + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}}$$

$$\therefore \qquad \varepsilon = \frac{\sqrt{1 + \left(\frac{2c.\omega}{c_c.\omega_n}\right)^2}}{\sqrt{\left(\frac{2c.\omega}{c_c.\omega_n}\right)^2 + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}} \qquad \ldots (i)$$

When the damper is not provided, then $c = 0$, and

$$\varepsilon = \frac{1}{1 - (\omega/\omega_n)^2} \qquad \ldots (ii)$$

From above, we see that when $\omega_n > 1$, ε is negative. This means that there is a phase difference of 180° between the transmitted force and the disturbing force ($F \cos\omega.t$). The value of ω/ω_n must be greater than √2 if ε is to be less than 1 and it is the numerical value of ε, independent of any phase difference between the forces that may exist which is important. It is therefore more convenient to use equation **(*ii*)** in the following form, *i.e.* $\varepsilon = \frac{1}{(\omega/\omega_{n})^{2} - 1}$

Fig.24 is the graph for different values of damping factor c/c_c to show the variation of transmissibility ratio (ε) against the ratio ω/ω_n. **1.** When $\omega/\omega_n = \sqrt{2}$, then all the curves pass through the point $\varepsilon = 1$ for all values of damping factor c/c_c

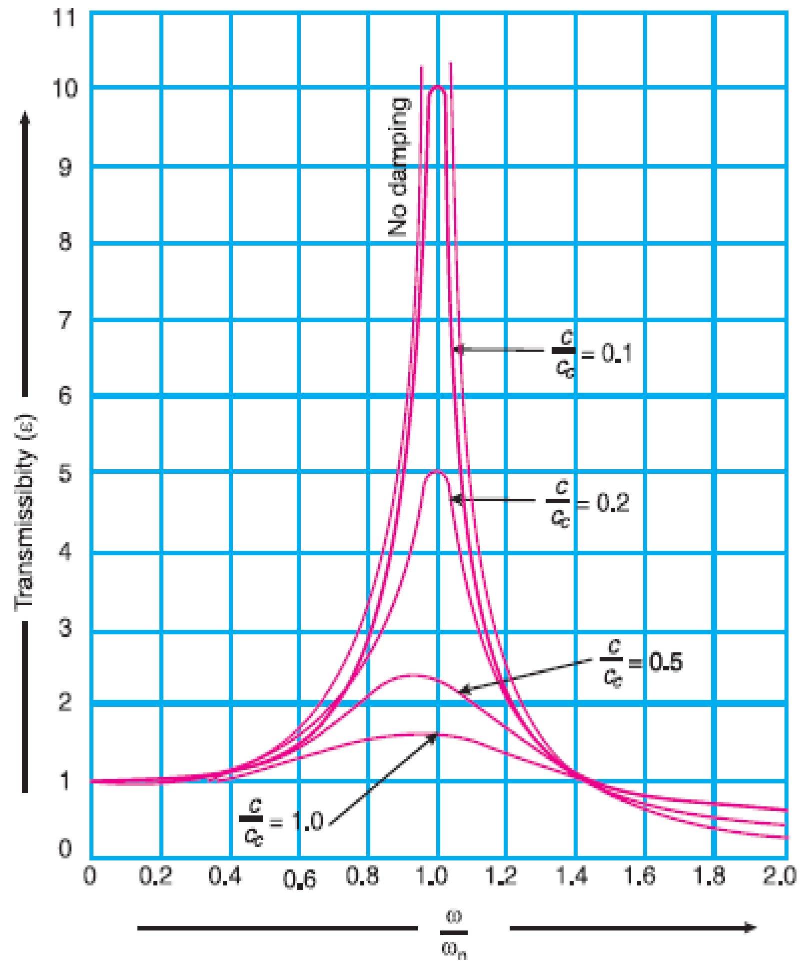


Fig.24. Graph showing the variation of transmissibility ratio.

2. When $\omega/\omega_n < 2$, then $\varepsilon > 1$ for all values of damping factor c/c_c. This means that the force transmitted to the foundation through elastic support is greater than the force applied.

3. When $\omega/\omega_n > 2$, then $\varepsilon < 1$ for all values of damping factor c/c_c. This shows that the force transmitted through elastic support is less than the applied force. Thus vibration isolation is possible only in the range of $\omega/\omega_n > 2$.

We also see from the curves in Fig.24 that the damping is detrimental beyond $\omega/\omega_n > 2$ and advantageous only in the region $\omega/\omega_n < 2$. It is thus concluded that for the vibration isolation, dampers need not to be provided but in order to limit resonance amplitude, stops may be provided.

EXERCISES (A)

1. A shaft of 100 mm diameter and 1 metre long is fixed at one end and other end carries a flywheel of mass 1 tonne. Taking Young's modulus for the shaft material as 200 GN/m2, find the natural frequency of longitudinal and transverse vibrations. **[Ans. 200 Hz; 8.6 Hz]**

2. A beam of length 10 m carries two loads of mass 200 kg at distances of 3 m from each end together with a central load of mass 1000 kg. Calculate the frequency of transverse vibrations. Neglect the mass of the beam and take I = 109 mm4 and E = 205×103 N/mm2. **[Ans. 13.8 Hz]**

3. A steel bar 25 mm wide and 50 mm deep is freely supported at two points 1 m apart and carries a mass of 200 kg in the middle of the bar. Neglecting the mass of the bar, find the frequency of transverse vibration.

If an additional mass of 200 kg is distributed uniformly over the length of the shaft, what will be the frequency of vibration? Take E = 200 GN/m2. **[Ans. 17.8 Hz; 14.6 Hz]**

4. A shaft 1.5 m long is supported in flexible bearings at the ends and carries two wheels each of 50 kg mass. One wheel is situated at the centre of the shaft and the other at a distance of 0.4 m from the centre towards right. The shaft is hollow of external diameter 75 mm and inner diameter 37.5 mm. The density of the shaft material is 8000 kg/m3. The Young's modulus for the shaft material is 200 GN/m2. Find the frequency of transverse vibration. **[Ans. 33.2 Hz]**

5. A shaft of diameter 10 mm carries at its centre a mass of 12 kg. It is supported by two short bearings, the centre distance of which is 400 mm. Find the whirling speed: 1.neglecting the mass of the shaft, and 2.taking the mass of the shaft also into consideration. The density of shaft material is 7500 kg/m3. **[Ans. 748 r.p.m.; 744 r.p.m.]**

6. A shaft 180 mm diameter is supported in two bearings 2.5 metres apart. It carries three discs of mass 250 kg, 500 kg and 200 kg at 0.6 m, 1.5 m and 2 m from the left hand. Assuming the mass of the shaft 190 kg/m, determine the critical speed of the shaft. Young's modulus for the material of the shaft is 211 GN/m2. **[Ans. 18.8 r.p.m.]**

7. A shaft 12.5 mm diameter rotates in long bearings and a disc of mass 16 kg is secured to a shaft at the middle of its length. The span of the shaft between the bearing is 0.5 m. The mass centre of the disc is 0.5 mm from the axis of the shaft. Neglecting the mass of the shaft and taking E = 200 GN/m2, find: 1 critical speed of rotation in r.p.m., and 2. the range of speed over which the stress in the shaft due to bending will not exceed 120 MN/m2. Take the static deflection of the shaft for a beam fixed at both ends, *i.e.* $\delta = \frac{wl^3}{192\ EI}$ **[Ans. 1450 r.p.m. ; 1184 to 2050 r.p.m.]**

8. A vertical shaft 25 mm diameter and 0.75 m long is mounted in long bearings and carries a pulley of mass 10 kg midway between the bearings. The centre of pulley is 0.5 mm from the axis of the shaft. Find (*a*) the whirling speed, and (*b*) the bending stress in the shaft, when it is rotating at 1700 r.p.m. Neglect the mass of the shaft and *E* = 200 GN/m2. **[Ans. 3996 r.p.m; 12.1 MN/m2]**

9. A shaft 12 mm in diameter and 600 mm long between long bearings carries a central mass of 4 kg.

If the centre of gravity of the mass is 0.2 mm from the axis of the shaft, compute the maximum flexural stress in the shaft when it is running at 90 per cent of its critical speed. The value of

Young's modulus of the material of the shaft is 200 GN/m2. **[Ans. 14.8 kN/m2]**

10. A vibrating system consists of a mass of 8 kg, spring of stiffness 5.6 N/mm and a dashpot of damping coefficient of 40 N/m/s. Find (*a*) damping factor, (*b*) logarithmic decrement, and (*c*) ratio of the two consecutive amplitudes. **[Ans. 0.094; 0.6; 1.82]**

11. A body of mass of 50 kg is supported by an elastic structure of stiffness 10 kN/m. The motion of the body is controlled by a dashpot such that the amplitude of vibration decreases to one-tenth of its original value after two complete vibrations. Determine: 1. the damping force at 1 m/s; 2. The damping ratio, and 3. the natural frequency of vibration. **[Ans. 252 N/m/s; 0.178; 2.214 Hz]**

12. A mass of 85 kg is supported on springs which deflect 18 mm under the weight of the mass. The vibrations of the mass are constrained to be linear and vertical and are damped by a dashpot which reduces the amplitude to one quarter of its initial value in two complete oscillations. Find: 1.The magnitude of the damping force at unit speed, and 2. the periodic time of damped vibration. **[Ans. 435 N/m/s; 0.27 s]**

EXERCISES (B)

1. What are the causes and effects of vibrations?

2. Define, in short, free vibrations, forced vibrations and damped vibrations.

3. Discuss briefly with neat sketches the longitudinal, transverse and torsional free vibrations.

4. Derive an expression for the natural frequency of free transverse and longitudinal vibrations by equilibrium method.

5. Discuss the effect of inertia of the shaft in longitudinal and transverse vibrations.

6. Deduce an expression for the natural frequency of free transverse vibrations for a simply supported shaft carrying uniformly distributed mass of *m* kg per unit length.
7. Deduce an expression for the natural frequency of free transverse vibrations for a beam fixed at both ends and carrying a uniformly distributed mass of *m* kg per unit length.
8. Establish an expression for the natural frequency of free transverse vibrations for a simply supported beam carrying a number of point loads, by (*a*) Energy method ; and (*b*) Dunkerley's method.
9. Explain the term 'whirling speed' or 'critical speed' of a shaft. Prove that the whirling speed for a rotating shaft is the same as the frequency of natural transverse vibration.
10. Derive the differential equation characterising the motion of an oscillation system subject to viscous damping and no periodic external force. Assuming the solution to the equation, find the frequency of oscillation of the system.
Explain the term 'Logarithmic decrement' as applied to damped vibrations.
11. Establish an expression for the amplitude of forced vibrations.
12. Explain the term 'dynamic magnifier'.
13. What do you understand by transmissibility?

TORSIONAL VIBRATION

3.1 Introduction

We have already discussed in the previous chapter that when the particles of a shaft or disc move in a circle about the axis of a shaft, then the vibrations are known as ***torsional vibrations.*** In this case, the shaft is twisted and untwisted alternately and torsional shear stresses are induced in the shaft. In this chapter, we shall now discuss the frequency of torsional vibrations of various systems.

Natural Frequency of Free Torsional Vibrations

Consider a shaft of negligible mass whose one end is fixed and the other end carrying a disc as shown in Fig. 25

Let θ = Angular displacement of the shaft from mean position after time t in radians,

m = Mass of disc in kg,

I = Mass moment of inertia of disc in kg-m^2 = $m.k^2$,

k = Radius of gyration in metres,

q = Torsional stiffness of the shaft in N-m.

$\therefore$ Restoring force $= q.\theta$ ... (*i*)

and accelerating force $= I \times \dfrac{d^2\theta}{dt^2}$ ... (*ii*)

Equating equations (*i*) and (*ii*), the equation of motion is

$$I \times \frac{d^2\theta}{dt^2} = -q.\theta$$

or

$$I \times \frac{d^2\theta}{dt^2} + q.\theta = 0$$

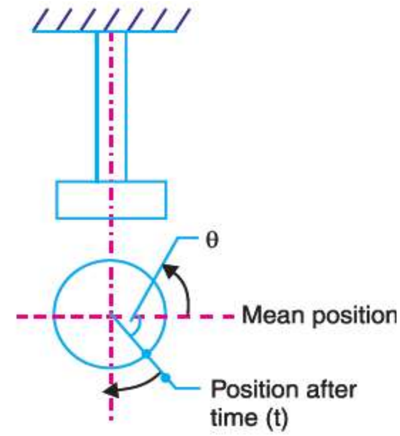


Fig 25. Natural frequency of free torsional vibrations.

$$\therefore \quad \frac{d^2\theta}{dt^2} = \frac{q}{I} \times \theta = 0 \qquad \text{..... (iii)}$$

The fundamental equation of the simple harmonic motion is

$$\frac{d^2\theta}{dt^2} + \omega^2 . X = 0 \qquad \text{..... (iv)}$$

Comparing equations (***iii***) and (***iv***), $\omega = \sqrt{\dfrac{q}{I}}$

$\therefore$ Time period, $t_p = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{q}}$

And natural frequency $f_n = \frac{1}{t_p} = \frac{1}{2\pi}\sqrt{\frac{q}{I}}$

Note : The value of the torsional stiffness q may be obtained from the torsion equation,

$$\frac{T}{J} = \frac{C.\theta}{l} \quad \text{or} \quad \frac{T}{\theta} = \frac{C.J}{l}$$

$$\therefore \quad q = \frac{C.J}{l} \quad \ldots \left(\because \frac{T}{\theta} = q\right)$$

where C = Modulus of rigidity for the shaft material,

J = Polar moment of inertia of the shaft cross-section,

$= \frac{\pi}{32} d^4$; d is the diameter of the shaft, and

l = Length of the shaft.

Example 3.1. *A shaft of 100 mm diameter and 1 metre long has one of its end fixed and the other end carries a disc of mass 500 kg at a radius of gyration of 450 mm. The modulus of rigidity for the shaft material is 80 GN/m^2. Determine the frequency of torsional vibrations.*

Solution. Given: d = 100 mm = 0.1 m; l = 1 m; m = 500 kg; k = 450 mm = 0.45m; C = 80 GN/m^2 = 80 × 109 N/m^2

We know that polar moment of inertia of the shaft,

$$J = \frac{\pi}{32} \times d^4 = \frac{\pi}{32}(0.1)^4 = 9.82 \times 10^{-6}\,\text{m}^4$$

$\therefore$ Torsional stiffness of the shaft,

$$q = \frac{C.J}{l} = \frac{80 \times 10^9 \times 9.82 \times 10^{-6}}{1} = 785.6 \times 10^3\ \text{N-m}$$

We know that mass moment of inertia of the shaft,

$$I = m.k^2 = 500\,(0.45)^2 = 101.25\ \text{kg-m}^2$$

$\therefore$ Frequency of torsional vibrations,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{q}{I}} = \frac{1}{2\pi}\sqrt{\frac{785.6 \times 10^3}{101.25}} = \frac{88.1}{2\pi} = 14\ \text{Hz}$$ **Ans.**

Example 3.2. *A flywheel is mounted on a vertical shaft as shown in Fig 24.2. The both ends of a shaft are fixed and its diameter is 50 mm. The flywheel has a mass of 500 kg and its radius of gyration is 0.5 m. Find the natural frequency of torsional vibrations, if the modulus of rigidity for the shaft material is 80 GN/m^2.*

Solution. Given: d = 50 mm = 0.05 m; m = 500 kg; k = 0.5m; G = 80 GN/m^2 = 84 × 109 N/m^2

We know that polar moment of inertia of the shaft,

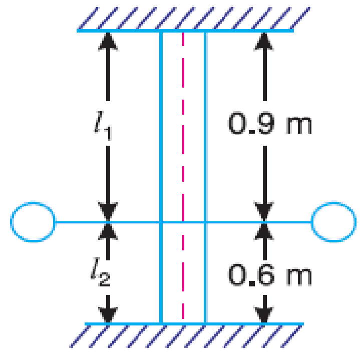


Fig. 26

$$J = \frac{\pi}{32} \times d^4 = \frac{\pi}{32}(0.05)^4 \text{ m}^4$$

$$= 0.6 \times 10^{-6} \text{ m}^4$$

∴ Torsional stiffness of the shaft for length l_1,

$$q_1 = \frac{C.J}{l_1} = \frac{84 \times 10^9 \times 0.6 \times 10^{-6}}{0.9}$$

$$= 56 \times 10^3 \text{ N-m}$$

Similarly torsional stiffness of the shaft for length l_2,

$$q_2 = \frac{C.J}{l_2} = \frac{84 \times 10^9 \times 0.6 \times 10^{-6}}{0.6} = 84 \times 10^3 \text{ N-m}$$

∴ Total torsional stiffness of the shaft,

$$q = q_1 + q_2 = 56 \times 10^3 + 84 \times 10^3 = 140 \times 10^3 \text{ N-m}$$

We know that mass moment of inertia of the flywheel,

$$I = m\,k^2 = 500\,(0.5)^2 = 125 \text{ kg-m}^2$$

∴ Natural frequency of torsional vibration,

$$f_n = \frac{1}{2\pi} = \sqrt{\frac{q}{I}} = \frac{1}{2\pi}\sqrt{\frac{140\times10^3}{125}} = \frac{33.5}{2\pi} = 5.32$$

3.2 Effect of Inertia of the Constraint on Torsional Vibrations

Consider a constraint *i.e.* shaft whose one end is fixed and the other end free, as shown in Fig.26.

Let ω = Angular velocity of free end,

m = Mass of constraint for unit length,

l = Length of constraint,

m_C = Total mass of constraint = $m.l$,

k = Radius of gyration of constraint,

I_C = Total mass moment of inertia of constraint

$= m_C.k^2 = m.l.k^2$.

Consider a small element at a distance x from the fixed end and of length δx. Therefore, Mass moment of inertia of the element

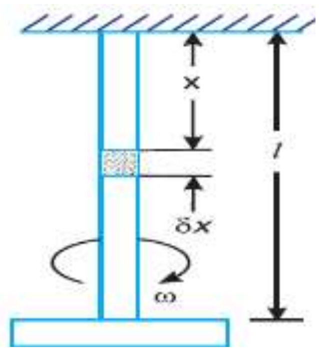


Fig 27. Effect of inertia of the constraint on torsional vibrations.

s

If a mass where mass moment of inertia is equal to IC/3 is placed at the free end and the constant is assumed to be of negligible mass, then the

Total Kinetic Energy of the constant

$$\frac{I}{2} \quad \frac{I_C}{3} \quad 2$$ ....[same as equation (i)]

Hence the two systems are dynamically same. Therefore the inertia of the constraint may be allowed for by adding $I_C/3$ to the mass moment of inertia I of the disc at the free end.

From the above discussion, we find that when the mass moment of inertia of the constraint I_C and the mass moment of inertia of the disc I are known, then natural frequency of vibration,

$$f_n \quad \frac{1}{2}\sqrt{\frac{q}{I \quad I_C/3}}$$

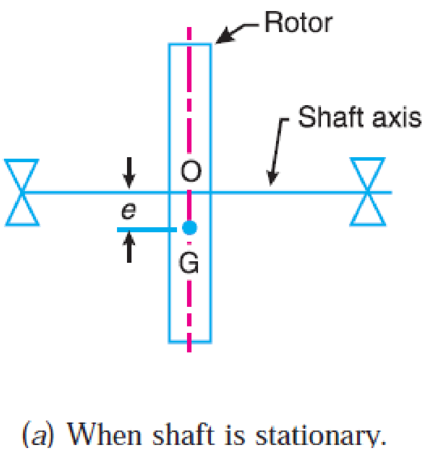
3.3. Critical or Whirling Speed of a Shaft

In actual practice, a rotating shaft carries different mountings and accessories in the form of gears, pulleys, etc. When the gears or pulleys are put on the shaft, the centre of gravity of the pulley or gear does not coincide with the centre line of the bearings or with the axis of the shaft, when the shaft is stationary. This means that the centre of gravity of the pulley or gear is at a certain distance from the axis of rotation and due to this, the shaft is subjected to centrifugal force.

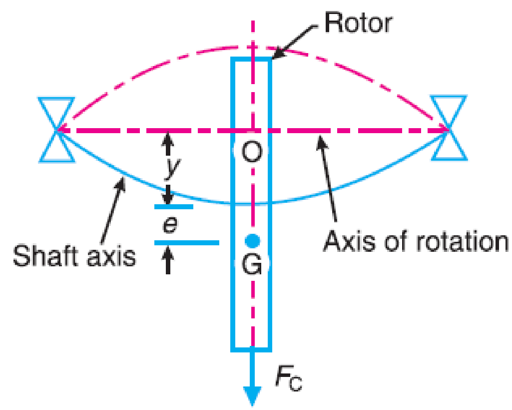
This force will bent the shaft which will further increase the distance of centre of gravity of the pulley or gear from the axis of rotation. This correspondingly increases the value of centrifugal force, which further increases the distance of centre of gravity from the axis of rotation. This effect is cumulative and ultimately the shaft fails. The bending of shaft not only depends upon the value of eccentricity (distance between centre of gravity of the pulley and the axis of rotation) but also depends upon the speed at which the shaft rotates.

The shaft continues to deflect infinitely from the axis of rotation at a critical speed called whirling speed.

The speed at which the shaft runs so that the additional deflection of the shaft from the axis of rotation becomes infinite, is known as *critical* or *whirling speed*.



(*a*) When shaft is stationary.



(*b*) When shaft is rotating.

Fig 28 Critical or whirling speed of a shaft.

Consider a shaft of negligible mass carrying a rotor, as shown in Fig.28 (*a*). The point *O* is on the shaft axis and *G* is the centre of gravity of the rotor. When the shaft is stationary, the centre line of the bearing and the axis of the shaft coincides. Fig. 28 (*b*) shows the shaft when rotating about the axis of rotation at a uniform speed of ω rad/s.

Let m = Mass of the rotor,

e = Initial distance of centre of gravity of the rotor from the centre line of the bearing or shaft axis, when the shaft is stationary,

y = Additional deflection of centre of gravity of the rotor when the shaft starts rotating at ω rad/s, and

s = Stiffness of the shaft *i.e.* the load required per unit deflection of the shaft.

Since the shaft is rotating at ω rad/s, therefore centrifugal force acting radially outwards through *G* causing the shaft to deflect is given by

$$F_C = m.\omega^2(y + e)$$

The shaft behaves like a spring. Therefore the force resisting the deflection,

$$Y = s.y$$

For the equilibrium position,

$$m.\omega^2 (y + e) = s.y$$

$$\text{or } m.\omega^2.y + m.\omega^2.e = s.y \text{ or } y(s - m.\omega^2) = m.\omega^2.e$$

$$\therefore \quad y = \frac{m.\omega^2.e}{s - m.\omega^2} = \frac{\omega^2.e}{\frac{s}{m} - \omega^2} \quad \text{...... (i)}$$

Recall, $\omega_n = \sqrt{\frac{s}{m}}$ or $y = \frac{\omega^2}{(\omega_n)^2 - \omega^2}$ ...{ from equation (i)}

If $\omega > \omega_n$ the deflection is negative but if $\omega_n > \omega$ the defection is positive.

$$\rightarrow \quad y = \mp \frac{\omega^2.e}{(\omega_n)^2 - \omega^2}$$

Divide both the numerator and the denominator by ω^2

$$\therefore \quad y = \mp \frac{e}{(\frac{\omega_n}{\omega})^2 - 1} \quad \text{(putting } \omega_n = c\text{)}$$

$$\rightarrow \quad y = \mp \frac{e}{(\frac{\omega_c}{\omega}) - 1}$$

When $\omega_n = \omega_c$, the value of y becomes infinite. Therefore ω_c is the critical or whirling speed.

Critical or whirling speed, $\omega_c = \omega_n = \sqrt{\frac{s}{m}}$

$$\rightarrow \quad \omega_c = \sqrt{\frac{g}{\partial}} \text{ HZ}$$

A little consideration will show that when $\omega > \omega_n$, the value of *y* will be negative and the shaft deflects is the opposite direction as shown dotted in Fig 28 (*b*).

In order to have the value of y always positive, both ***plus*** and ***minus*** signs are taken.

$$\therefore \quad y = \pm \frac{\omega^2 e}{(\omega_n)^2 - \omega^2} = \frac{\pm e}{\left(\frac{\omega_n}{\omega}\right)^2 - 1} = \frac{\pm e}{\left(\frac{\omega_c}{\omega}\right)^2 - 1}$$

... (Substituting $\omega_n = \omega_c$)

We see from the above expression that when $\omega_n = \omega_c$, the value of y becomes infinite. Therefore ω_c is the **critical or whirling speed.**

$\therefore$ Critical or whirling speed,

$$\omega_c = \omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{g}{\delta}} \text{ Hz} \quad \ldots \left(\because \ \delta = \frac{m.g}{s}\right)$$

If N_c is the critical or whirling speed in r.p.s., then

$$2\pi N_c = \sqrt{\frac{g}{\delta}} \quad \text{or} \quad N_c = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} = \frac{0.4985}{\sqrt{\delta}} \text{ r.p.s.}$$

where δ = Static deflection of the shaft in metres.

Hence the **critical or whirling speed is the same as the natural frequency of transverse vibration but its unit will be revolutions per second.**

Notes: 1. When the centre of gravity of the rotor lies between the centre line of the shaft and the centre line of the bearing, e is taken negative. On the other hand, if the centre of gravity of the rotor does not lie between the centre line of the shaft and the centre line of the bearing (as in the above article) the value of e is taken positive.

2. To determine the critical speed of a shaft which may be subjected to point loads, uniformly distributed load or combination of both, find the frequency of transverse vibration which is equal to critical speed of a shaft in r.p.s. The Dunkerley's method may be used for calculating the frequency.

3. A shaft supported is short bearings (or ball bearings) is assumed to be a simply supported shaft while the shaft supported in long bearings (or journal bearings) is assumed to have both ends fixed.

A shaft is said to whirl when it rotates in a bowed condition between the bearings under certain conditions if the speed of rotation of a shaft is gradually increased from zero, it will be seen that approaching a particular speed the shaft will begin to bow, until the bowing is maximum at that speed. At the further increase of the speed, the bow disappears.

The speed at which the shaft runs, so that the bowed condition (deflection) is a maximum is known as critical or whirling speed; and they represent a dangerous condition which needs to be avoided. In spite of careful balancing since whirling cannot be avoided then, the critical speed needs to be identified. The equipment employing lengths of shafting and high speed are particularly prone to the problem. Turbine rotors and shafts are two examples.

The need for whirling originate in the need to provide centripetal acceleration for a mass to rotate in a circular path. This will arise when the centre of gravity of a rotating mass does not coincide with the axis of the rotation. For a shaft this may be due to:

(i) Imperfect machining,
(ii) Deflection due to the weight of shaft,
(iii) Deflection due to a mass attached to the shaft etc.

Whatsoever the cause might be, it is the strength of the shaft material which must provide the centripetal force to give the required acceleration. At the critical or whirling speeds, the elastic stiffness of the shaft is just able to provide the necessary centripetal force and a condition of unstable equilibrium is reached where failure may eventually result.

Example 3.3. *Calculate the whirling speed of a shaft 20 mm diameter and 0.6 m long carrying a mass of 1 kg at its mid-point. The density of the shaft material is 40 Mg/m^3, and Young's modulus is 200 GN/m^2. Assume the shaft to be freely supported.*

Solution. Given: d = 20 mm = 0.02 m; l = 0.6 m; m_1 = 1 kg ; ρ = 40 Mg/m^3 = 40 × 106 g/m^3 = 40 × 103 kg/m^3; E = 200 GN/m^2 = 200 × 109 N/m^2

We know that moment of inertia of the shaft,

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64}(0.02)^4 \text{ m}^4$$
$$= 7.855 \times 10^{-9} \text{ m}^4$$

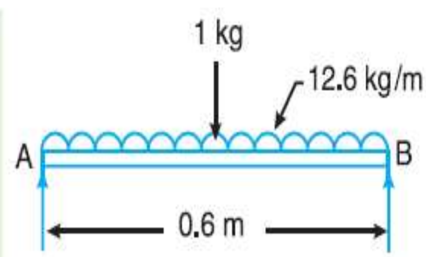


Fig. 29

Since the density of shaft material is 40 × 10^3 kg/m^3, therefore mass of the shaft per metre length,

$$m_S = \text{Area} \times \text{length} \times \text{density} = \frac{\pi}{4}(0.02)^2 \times 1 \times 40 \times 10^3 = 12.6 \text{ kg/m}$$

We know that static deflection due to 1 kg of mass at the centre,

$$\delta = \frac{Wl^3}{48\,EI} = \frac{1 \times 9.81\,(0.6)^3}{48 \times 200 \times 10^9 \times 7.855 \times 10^{-9}} = 28 \times 10^{-6} \text{ m}$$

and static deflection due to mass of the shaft,

$$\delta_S = \frac{5\,wl^4}{384\,EI} = \frac{5 \times 12.6 \times 9.81\,(0.6)^4}{384 \times 200 \times 10^9 \times 7.855 \times 10^{-9}} = 0.133 \times 10^{-3} \text{ m}$$

$\therefore$ Frequency of transverse vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta + \dfrac{\delta_S}{1.27}}} + \frac{0.4985}{\sqrt{28 \times 10^{-6} + \dfrac{0.133 \times 10^{-3}}{1.27}}}$$

$$= \frac{0.4985}{11.52 \times 10^{-3}} = 43.3 \text{ Hz}$$

Let N_c = Whirling speed of a shaft.

We know that whirling speed of a shaft in r.p.s. is equal to the frequency of transverse vibration in Hz , therefore

$$N_c = 43.3 \text{ r.p.s.} = 43.3 \times 60 = 2598 \text{ r.p.m.}$$ **Ans.**

Example 3.4. *A shaft 1.5 m long, supported in flexible bearings at the ends carries two wheels each of 50 kg mass. One wheel is situated at the centre of the shaft and the other at a distance of 375 mm from the centre towards left. The shaft is hollow of external diameter 75 mm and internal diameter 40 mm. The density of the shaft material is 7700 kg/m3 and its modulus of elasticity is 200 GN/m2. Find the lowest whirling speed of the shaft, taking into account the mass of the shaft.*

Solution. **Given:** l = 1.5 m; $m_1 = m_2$ = 50 kg; d_1 = 75 mm = 0.075 m; $d2$ = 40 mm = 0.04 m; ρ = 7700 kg/m^3; E = 200 GN/m^2 = 200 × 109 N/m^2

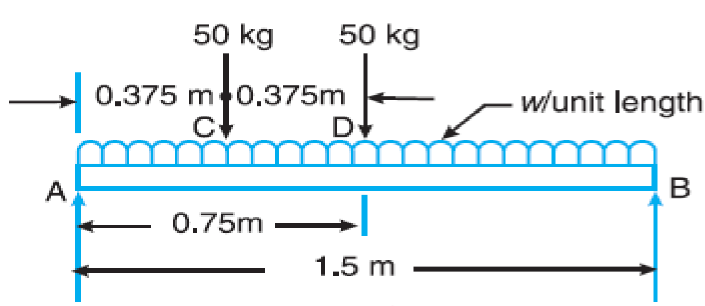


Fig 30

We know that moment of inertia of the shaft,

We know that static deflection due to uniformly distributed load or mass of the shaft,

$$\delta_S = \frac{5}{384} \times \frac{wl^4}{EI} = \frac{5}{384} \times \frac{24.34 \times 9.81\,(1.5)^4}{200 \times 10^9 \times 1.4 \times 10^{-6}} = 56 \times 10^{-6} \text{ m}$$

. . . (Substituting, $w = m_S \times g$)

We know that frequency of transverse vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta_1 + \delta_2 + \frac{\delta_S}{1.27}}} = \frac{0.4985}{\sqrt{70 \times 10^{-6} + 123 \times 10^{-6} + \frac{56 \times 10^{-6}}{1.27}}} \text{ Hz}$$

$$= 32.4 \text{ Hz}$$

Since the whirling speed of shaft (N_c) in r.p.s. is equal to the frequency of transverse vibration in Hz, therefore

$$N_c = 32.4 \text{ r.p.s.} = 32.4 \times 60 = 1944 \text{ r.p.m.}$$ **Ans.**

$$I = \frac{\pi}{64}\left[(d_1)^4 - (d_2)^4\right] = \frac{\pi}{64}\left[(0.075)^4 - (0.04)^4\right] = 1.4 \times 10^{-6} \text{ m}^4$$

Since the density of shaft material is 7700 kg/m^3, therefore mass of the shaft per metre length,

$$m_S = \text{Area} \times \text{length} \times \text{density}$$

$$= \frac{\pi}{4}\left[(0.075)^2 - (0.04)^2\right] 1 \times 7700 = 24.34 \text{ kg/m}$$

We know that the static deflection due to a load W

$$= \frac{Wa^2b^2}{3\,EIl} = \frac{m.ga^2b^2}{3\,EIl}$$

$\therefore$ Static deflection due to a mass of 50 kg at C,

$$\delta_1 = \frac{m_1 ga^2b^2}{3\,EIl} = \frac{50 \times 9.81\,(0.375)^2\,(1.125)^2}{3 \times 200 \times 10^9 \times 1.4 \times 10^{-6} \times 1.5} = 70 \times 10^{-6} \text{ m}$$

. . . (Here a = 0.375 m, and b = 1.125 m)

Similarly, static deflection due to a mass of 50 kg at D

$$\delta_2 = \frac{m_1 ga^2b^2}{3\,EIl} = \frac{50 \times 9.81\,(0.75)^2\,(0.75)^2}{3 \times 200 \times 10^9 \times 1.4 \times 10^{-6} \times 1.5} = 123 \times 10^{-6} \text{ m}$$

. . . (Here $a = b$ = 0.75 m)

We know that static deflection due to uniformly distributed load or mass of the shaft,

$$\delta_S = \frac{5}{384} \times \frac{wl^4}{EI} = \frac{5}{384} \times \frac{24.34 \times 9.81 (1.5)^4}{200 \times 10^9 \times 1.4 \times 10^{-6}} = 56 \times 10^{-6} \text{ m}$$

. . . (Substituting, $w = m_S \times g$)

We know that frequency of transverse vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta_1 + \delta_2 + \frac{\delta_S}{1.27}}} = \frac{0.4985}{\sqrt{70 \times 10^{-6} + 123 \times 10^{-6} + \frac{56 \times 10^{-6}}{1.27}}} \text{ Hz}$$

$$= 32.4 \text{ Hz}$$

Since the whirling speed of shaft (N_c) in r.p.s. is equal to the frequency of transverse vibration in Hz, therefore

$$N_c = 32.4 \text{ r.p.s.} = 32.4 \times 60 = 1944 \text{ r.p.m.}$$ **Ans.**

Example 3.5. *A vertical shaft of 5 mm diameter is 200 mm long and is supported in long bearings at its ends. A disc of mass 50 kg is attached to the centre of the shaft. Neglecting any increase in stiffness due to the attachment of the disc to the shaft, find the critical speed of rotation and the maximum bending stress when the shaft is rotating at 75% of the critical speed. The centre of the disc is 0.25 mm from the geometric axis of the shaft. E = 200 GN/m^2.*

Solution. Given: d = 5 mm = 0.005 m; l = 200 mm = 0.2 m; m = 50 kg; e = 0.25 mm = 0.25×10^{-3} m ; E = 200 GN/m^2 = 200 × 109 N/m^2

Critical speed of rotation

We know that moment of inertia of the shaft,

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64} (0.005)^4 = 30.7 \times 10^{-12} \text{m}^4$$

Since the shaft is supported in long bearings, it is assumed to be fixed at both ends. We know that the static deflection at the centre of the shaft due to a mass of 50 kg,

$$\delta = \frac{Wl^3}{192 EI} = \frac{50 \times 9.81 (0.2)^3}{192 \times 200 \times 10^9 \times 30.7 \times 10^{-12}} = 3.33 \times 10^{-3} \text{m}$$

. . . ($\because$ $W = m.g$)

We know that critical speed of rotation (or natural frequency of transverse vibrations),

$$N_c = \frac{0.4985}{\sqrt{3.33 \times 10^{-3}}} = 8.64 \text{ r.p.s.}$$ **Ans.**

Maximum bending stress

Let $\boldsymbol{\sigma}$ = Maximum bending stress in N/m2, and

N = Speed of the shaft = 75% of critical speed = 0.75 N_c . . . (Given)

When the shaft starts rotating, the additional dynamic load (W_1) to which the shaft is subjected, may be obtained by using the bending equation,

$$\frac{M}{I} = \frac{\sigma}{y} \text{ or } M = \frac{\sigma . I}{y1}$$

We know that for a shaft fixed at both ends and carrying a point load (W_1) at the centre, the maximum bending moment

$$M = \frac{W_1 . l}{8}$$

$$\therefore \quad \frac{W_1 . l}{8} = \frac{\sigma \cdot I}{d/2} \qquad \ldots (\because\ y_1 = d/2)$$

and
$$W_1 = \frac{\sigma . I}{d/2} \times \frac{8}{l} = \frac{\sigma \times 30.7 \times 10^{-12}}{0.005/2} \times \frac{8}{0.2} = 0.49 \times 10^{-6} \sigma \, \text{N}$$

$\therefore$ Additional deflection due to load W_1,

$$y = \frac{W_1}{W} \times \delta = \frac{0.49 \times 10^{-6} \sigma}{50 \times 9.81} \times 3.33 \times 10^{-3} = 3.327 \times 10^{-12} \sigma$$

We know that

$$y = \frac{\pm e}{\left(\frac{\omega_c}{\omega}\right)^2 - 1} = \frac{\pm e}{\left(\frac{N_c}{N}\right) - 1} \qquad \ldots \text{(Substituting } \omega_c = N_c \text{ and } \omega = N\text{)}$$

$$3.327 \times 10^{-12} \sigma = \frac{\pm 0.25 \times 10^{-3}}{\left(\frac{N_c}{0.75\, N_c}\right)^2 - 1} = \pm 0.32 \times 10^{-3}$$

$\sigma = 0.32 \times 10^{-3} / 3.327 \times 10^{-12} = 0.0962 \times 10^{9}$ N/m^2 ….(Taking + ve sign)
= 96.2×10^6 N/m^2 = 96.2 MN/m^2 **Ans.**

Example 3.6. *A vertical steel shaft 15 mm diameter is held in long bearings 1 metre apart and carries at its middle a disc of mass 15 kg. The eccentricity of the centre of gravity of the disc from the centre of the rotor is 0.30 mm. The modulus of elasticity for the shaft material is 200 GN/m^2 and the permissible stress is 70 MN/m^2. Determine:* **1.** *The critical speed of the shaft and* **2.** *The range of speed over which it is unsafe to run the shaft. Neglect the mass of the shaft.*

[*For a shaft with fixed end carrying a concentrated load (W) at the centre assume*

$\delta = \frac{Wl^3}{192EI}$ and M=$\frac{W.l}{8}$, whereδ, *and M are maximum deflection and bending moment respectively*].

Solution. Given: d = 15 mm = 0.015 m; l = 1 m; m = 15 kg; e = 0.3 mm = 0.3 × 10^{-3} m; E = 200 GN/m^2 = 200 × 10^9 N/m^2; $\boldsymbol{\sigma}$ = 70 MN/m^2 = 70 × 10^6 N/m^2

We know that moment of inertia of the shaft,

$$I = \frac{\pi\, d^4}{64} = \frac{\pi}{64}\,(0.015)^4$$
$$= 2.5\times10^{-9}\ \text{m}^4$$

1. *Critical speed of the shaft*

Since the shaft is held in long bearings, therefore it is assumed to be fixed at both ends. We know that the static deflection at the centre of shaft,

$$\delta = \frac{Wl^3}{192EI} = \frac{15\times9.81\times1^3}{192\times200\times10^9\times2.5\times10^{-9}}$$
$$= 1.5\times10^{-3}\ \text{m} \qquad (\because\ W = mg)$$

$\therefore$ Natural frequency of transverse vibrations,

$$f_n = \frac{0.4985}{\sqrt{\delta}} = \frac{0.4985}{\sqrt{1.5\times10^{-3}}} = 12.88\,\text{Hz}$$

We know that the critical speed of the shaft in r.p.s. is equal to the natural frequency of transverse vibrations in Hz.

$\therefore$ Critical speed of the shaft,

$$N_c = 12.88 \text{ r.p.s.} = 12.88\times60 = 772.8 \text{ r.p.m.}\ \textbf{Ans.}$$

2. *Range of speed*

Let N_1 and N_2 = Minimum and maximum speed respectively.

When the shaft starts rotating, the additional dynamic load ($W_1 = m_1.g$) to which the shaft is subjected may be obtained from the relation

$$\frac{M}{I} = \frac{\sigma}{y_1} \quad \text{or} \quad M = \frac{\sigma.I}{y_1}$$

Since $M = \frac{W_1.l}{8} = \frac{m_1.g.l}{8}$, and $y_1 = \frac{d}{2}$, therefore

$$\frac{m_1.g.l}{8} = \frac{\sigma.I}{d/2}$$

or $$m_1 = \frac{8 \times 2 \times \sigma \times I}{d.g.l} = \frac{8 \times 2 \times 70 \times 10^6 \times 2.5 \times 10^{-9}}{0.015 \times 9.81 \times 1} = 19 \text{ kg}$$

$\therefore$ Additional deflection due to load $W_1 = m_1 g$,

$$y = \frac{W_1}{W} \times \delta = \frac{m_1}{m} \times \delta = \frac{19}{15} \times 1.5 \times 10^{-3} = 1.9 \times 10^{-3} \text{ m}$$

We know that,

$$y = \frac{\pm e}{\left(\frac{\omega_c}{\omega}\right)^2 - 1} \quad \text{or} \quad \pm \frac{y}{e} = \frac{1}{\left(\frac{N_c}{N}\right)^2 - 1}$$

. . . (Substituting, $\omega_c = N_c$, and $\omega = N$)

$$\therefore \quad \pm \frac{1.9 \times 10^{-3}}{0.3 \times 10^{-3}} = \frac{1}{\left(\frac{N_c}{N}\right)^2 - 1} \quad \text{or} \quad \left(\frac{N_c}{N}\right)^2 - 1 = \pm \frac{0.3}{1.9} = \pm 0.16$$

$$\left(\frac{N_c}{N}\right)^2 = 1 \pm 0.16 = 1.16 \text{ or } 0.84$$

. . . (Taking first plus sign and then negative sign)

Or $$N = \frac{N_C}{\sqrt{1.16}} = or \ \frac{N_C}{\sqrt{0.84}}$$

$$\therefore \quad N_1 = \frac{N_C}{\sqrt{1.16}} = \frac{772.8}{\sqrt{1.16}}$$

= 7.18 r.p.m

And $$N_2 = \frac{N_C}{\sqrt{0.84}} = \frac{772.8}{\sqrt{0.84}}$$

= 843 r.p.m

Hence range of speed is from 718 r.p.m to 843 r.p.m

3.4 Balancing of Rotating Masses

3.4.1 Introduction

The high speed of engines and other machines is a common phenomenon now-a-days. It is, therefore, very essential that all the rotating and reciprocating parts should be completely balanced as far as possible. If these parts are not properly balanced, the dynamic forces are set up. These forces not only increase the loads on bearings and stresses in the various members, but also produce unpleasant and even dangerous vibrations. In this chapter we shall discuss the balancing of unbalanced forces caused by rotating masses, in order to minimize pressure on the main bearings when an engine is running.

3.4.2. Balancing of Rotating Masses

We have already discussed, that whenever a certain mass is attached to a rotating shaft, it exerts some centrifugal force, whose effect is to bend the shaft and to produce vibrations in it. In order to prevent the effect of centrifugal force, another mass is attached to the opposite side of the shaft, at such a position so as to balance the effect of the centrifugal force of the first mass. This is done in such a way that the centrifugal force of both the masses are made to be equal and opposite. The process of providing the second mass in order to counteract the effect of the centrifugal force of the first mass, is called ***balancing of rotating masses.***

3.4.3 Balancing of a Single Rotating Mass By a Single Mass Rotating in the Same Plane

Consider a disturbing mass $m1$ attached to a shaft rotating at ω rad/s as shown in Fig. 31.

Let r_1 be the radius of rotation of the mass m_1 (*i.e.* distance between the axis of rotation of the shaft and the centre of gravity of the mass m_1).

We know that the centrifugal force exerted by the mass m_1 on the shaft,

$$F_{C1} = m_1.\omega^2.\ r_1 \quad \ldots \textbf{(\textit{i})}$$

This centrifugal force acts radially outwards and thus produces bending moment on the shaft. In order to counteract the effect of this force, a balancing mass ($m2$) may be attached in the same plane of rotation as that of disturbing mass ($m1$) such that the centrifugal forces due to the two masses are equal and opposite.

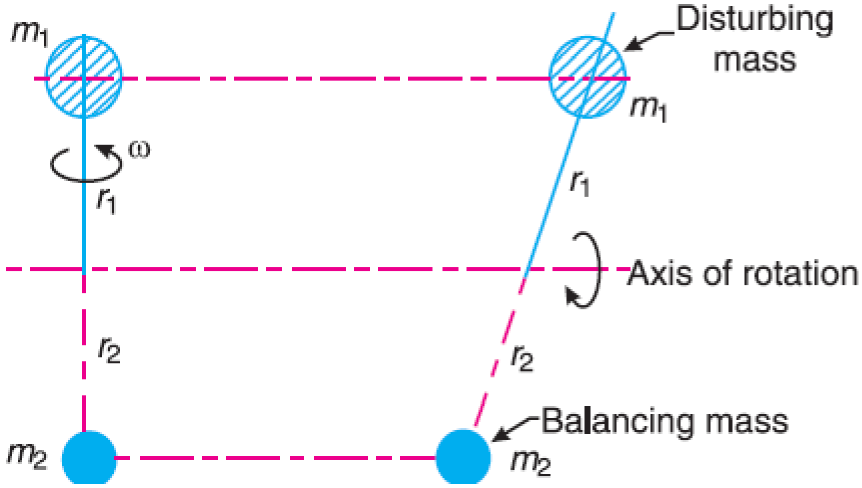


Fig. 31 Balancing of a single rotating mass by a single mass rotating in the same plane.

Let r_2 = Radius of rotation of the balancing mass m_2 (*i.e.* distance between the axis of rotation of the shaft and the centre of gravity of mass m_2).

∴ Centrifugal force due to mass m_2,

$$F_{C2} = m_2 .\ \omega^{2.}\ r_2 \ . \ . \ . \ (ii)$$

Equating equations **(*i*)** and **(*ii*),**

$$m_1.\omega_{1.}\ r_1 = m_2.\ \omega_2.\ r_2 \text{ or } m_1.\ r_1 = m_2.r_2$$

Notes: 1 The product $m_2.r_2$ may be split up in any convenient way. But the radius of rotation of the balancing mass (m_2) is generally made large in order to reduce the balancing mass m_2.

2. The centrifugal forces are proportional to the product of the mass and radius of rotation of respective masses, because ω^2 is same for each mass.

3.4.4 Balancing of a Single Rotating Mass By Two Masses Rotating in Different Planes

We have discussed in the previous article that by introducing a single balancing mass in the same plane of rotation as that of disturbing mass, the centrifugal forces are balanced. In other words, the two forces are equal in magnitude and opposite in direction. But this type of arrangement for balancing gives rise to a couple which tends to rock the shaft in its bearings. Therefore in order to put the system in complete balance, two balancing masses are placed in two different planes, parallel to the plane of rotation of the disturbing mass, in such a way that they satisfy the following two conditions of equilibrium.

1. The net dynamic force acting on the shaft is equal to zero. This requires that the line of action of three centrifugal forces must be the same. In other words, the centre of the masses of the system must lie on the axis of rotation. This is the condition for ***static balancing*.**

2. The net couple due to the dynamic forces acting on the shaft is equal to zero. In other words, the algebraic sum of the moments about any point in the plane must be zero.

The conditions **(1)** and **(2)** together give ***dynamic balancing*.** The following two possibilities may arise while attaching the two balancing masses:

1. The plane of the disturbing mass may be in between the planes of the two balancing masses, and

2. The plane of the disturbing mass may lie on the left or right of the two planes containing the balancing masses.

We shall now discuss both the above cases one by one.

1. When the plane of the disturbing mass lies in between the planes of the two balancing masses

Consider a disturbing mass m lying in a plane A to be balanced by two rotating masses m_1 and m_2 lying in two different planes L and M as shown in Fig. 31 Let r, r_1 and r_2 be the radii of rotation of the masses in planes A, L and M respectively.

Let l_1 = Distance between the planes A and L,

l_2 = Distance between the planes A and M, and

l = Distance between the planes L and M.

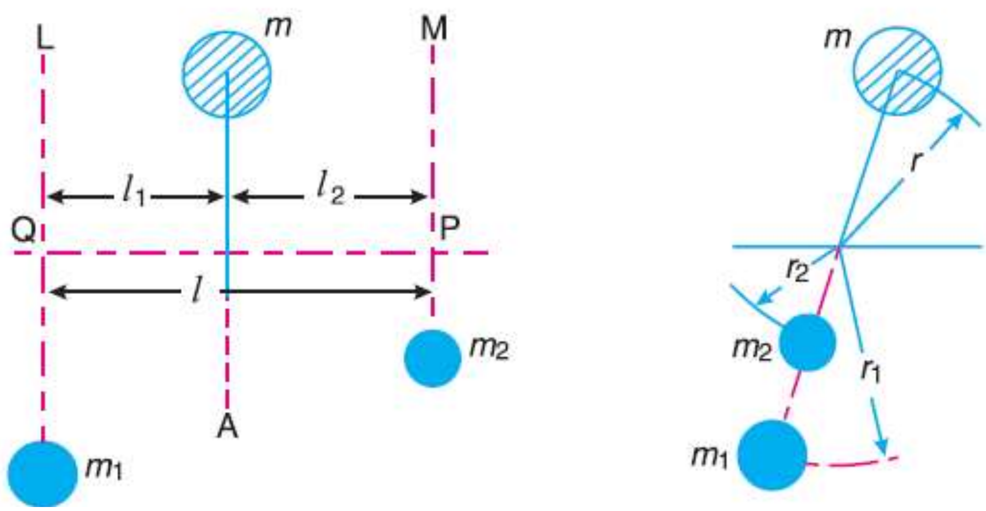


Fig. 32. Balancing of a single rotating mass by two rotating masses in different planes when the plane of single rotating mass lies in between the planes of two balancing masses. We know that the centrifugal force exerted by the mass m in the plane A,

$$F_C = m.\ \omega^2.\ r$$

Similarly, the centrifugal force exerted by the mass $m1$ in the plane L,

$$F_{C1} = m_1.\ \omega^2.\ r_1$$

and, the centrifugal force exerted by the mass $m2$ in the plane M,

$$F_{C2} = m_2.\ \omega^2.\ r_2$$

Since the net force acting on the shaft must be equal to zero, therefore the centrifugal force on the disturbing mass must be equal to the sum of the centrifugal forces on the balancing masses, therefore

$$FC = F_{C1} + F_{C2} \text{ or } m.\ \omega^2.r = m_1.\omega^2.r_1 + m_2\ .\ \omega^2\ r_2$$

$$\therefore \quad m\cdot r = m_1 \cdot r_1 + m_2 \cdot r_2 \qquad \text{. . . } (i)$$

Now in order to find the magnitude of balancing force in the plane L (or the dynamic force at the bearing Q of a shaft), take moments about P which is the point of intersection of the plane M and the axis of rotation. Therefore

$$F_{C1} \times l = F_C \times l_2 \text{ or } m_1.\omega^2.r_1\times l = m.\omega^2\times r\times l_2$$

$$\therefore \quad m_1\times r_1\times l = m.r.l_2 \text{ or } m_1\times r_1 = \text{m.r} \times \frac{l_2}{l} \qquad \text{. . . . } (ii)$$

Similarly, in order to find the balancing force in plane M (or the dynamic force at the bearing P of a shaft), take moments about Q which is the point of intersection of the plane L and the axis of rotation. Therefore

$$F_{C2} \times l = F_C\times l_1 \text{ or } m_2\ .\omega^2.\ r_2 \times l$$
$$= m\ .\omega^2.\ r \times l_1$$

$$\therefore \quad m_2 \cdot r_2 \times l = m.r.l_1 \text{ or } \text{m}_2.\text{r}_2 = \text{m.r} \times \frac{l_1}{l} \qquad \text{. . . . } (iii)$$

It may be noted that equation ***(i)*** represents the condition for static balance, but in order to achieve dynamic balance, equations ***(ii)*** or ***(iii)*** must also be satisfied.

2. When the plane of the disturbing mass lies on one end of the planes of the balancing masses

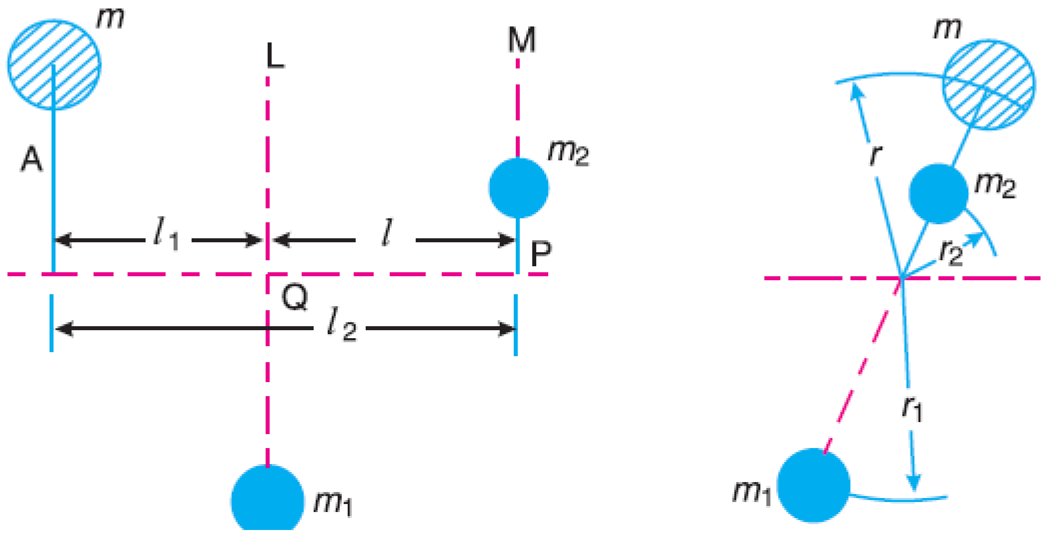


Fig 33

Fig.33 sows the balancing of a single rotating mass by two rotating masses in different planes, when the plane of single rotating mass lines at one end of the planes of balancing masses.

In this case, the mass *m* lies in the plane *A* and the balancing masses lie in the planes *L* and *M*, as shown in Fig.33. As discussed above, the following conditions must be satisfied in order to balance the system, *i.e*.

$$F_C + F_{C2} = F_{C1} \quad \text{or} \quad m \cdot \omega^2 \cdot r + m_2 \cdot \omega^2 \cdot r_2 = m_1 \cdot \omega^2 \cdot r_1$$

$$\therefore \quad m \cdot r + m_2 \cdot r_2 = m_1 . r_1 \quad \ldots (iv)$$

Now, to find the balancing force in the plane *L* (or the dynamic force at the bearing *Q* of a shaft), take moments about *P* which is the point of intersection of the plane *M* and the axis of rotation. Therefore

$$F_{C1} \times l = F_C \times l_2 \quad \text{or} \quad m_1 \cdot \omega^2 \cdot r_1 \times l = m \cdot \omega^2 \cdot r \times l_2$$

$$\therefore \quad m_1 \cdot r_1 \cdot l = m \cdot r \cdot l_2 \quad \text{or} \quad m_1 \cdot r_1 = m \cdot r \times \frac{l_2}{l} \quad \ldots (v)$$

. . . [Same as equation (*ii*)]

Similarly, to find the balancing force in the plane *M* (or the dynamic force at the bearing *P* of a shaft), take moments about *Q* which is the point of intersection of the plane *L* and the axis of rotation. Therefore

$$F_{C2} \times l = F_C \times l_1 \quad \text{or} \quad m_2 \cdot \omega^2 \cdot r_2 \times l = m \cdot \omega^2 \cdot r \times l_1$$

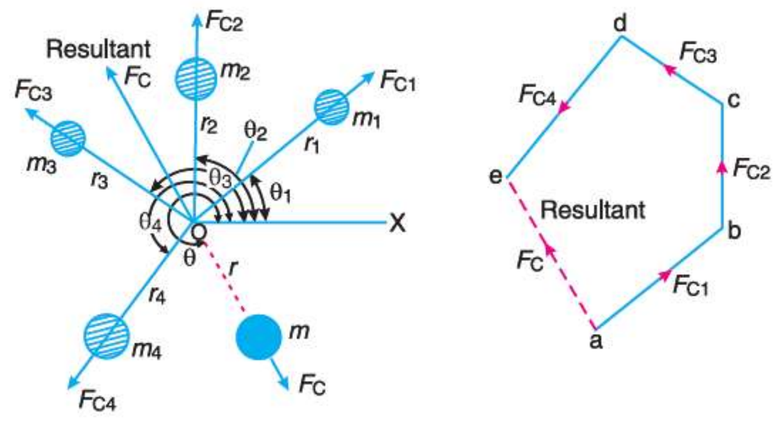
$$m_2 \cdot r_2 \cdot l = m \cdot r \cdot l_1 \quad \text{or} \quad m_2 \cdot r_2 = m \cdot r \times \frac{l_1}{l} \quad \ldots (vi)$$

. . . [Same as equation (*iii*)]

3.4.5. Balancing of Several Masses Rotating in the Same Plane

Consider any number of masses (say four) of magnitude m_1, m_2, m_3 and m_4 at distances of r_1, r_2, r_3 and r_4 from the axis of the rotating shaft. Let θ_1, θ_2, θ_3 and θ_4 be the angles of these masses with the horizontal line *OX*, as shown in Fig.34 (*a*). Let these masses rotate about an axis through *O* and perpendicular to the plane of paper, with a constant angular velocity of ωrad/s.

The magnitude and position of the balancing mass may be found out analytically or graphically as discussed below :



(a) Space diagram. (b) Vector diagram.

Fig 34 Balancing of several masses rotating in the same plane.

1. *Analytical method*

The magnitude and direction of the balancing mass may be obtained, analytically, as discussed below:

1. First of all, find out the centrifugal force* (or the product of the mass and its radius of rotation) exerted by each mass on the rotating shaft.

* Since ω^2 is same for each mass, therefore the magnitude of the centrifugal force for each mass is proportional to the product of the respective mass and its radius of rotation.

2. Resolve the centrifugal forces horizontally and vertically and find their sums, *i.e.* ΣH and ΣV We know that

Sum of horizontal components of the centrifugal forces,

$$\Sigma H = m_1.r_1 \cos\theta_1 + m_2 .r_2 \cos\theta_2 + \ldots\ldots$$

and sum of vertical components of the centrifugal forces,

$$\Sigma V = m_1 \cdot r_1 \sin \theta_1 + m_2 \cdot r_2 \sin\theta_2 + \ldots\ldots$$

3 Magnitude of the resultant centrifugal force,

$$F_C = \sqrt{(\Sigma H)^2 + (\Sigma V)^2}$$

4. If θ is the angle, which the resultant force makes with the horizontal, then

$$\tan \theta = \Sigma V / \Sigma H$$

5. The balancing force is then equal to the resultant force, but in ***opposite direction***.

6. Now find out the magnitude of the balancing mass, such that

$$F_C = m.\ r$$

where m = Balancing mass, and r = Its radius of rotation.

Example 3.7 *Four masses m_1, m_2, m_3 and m_4 are 200 kg, 300 kg, 240 kg and 260 kg respectively. The corresponding radii of rotation are 0.2 m, 0.15 m, 0.25 m and 0.3 m respectively and the angles between successive masses are 45°, 75° and 135°. Find the position and magnitude of the balance mass required, if its radius of rotation is 0.2 m*

.

Solution. Given: $m_1 = 200$ kg; $m_2 = 300$ kg ; $m_3 = 240$ kg ; $m_4 = 260$ kg ; $r_1 = 0.2$ m ; $r_2 = 0.15$ m ; $r_3 = 0.25$ m ; $r_4 = 0.3$ m ; $\theta_1 = 0°$; $\theta_2 = 45°$; $\theta_3 = 45° + 75° = 120°$; $\theta_4 = 45° + 75° + 135° = 255°$; $r = 0.2$ m

Let m = Balancing mass, and

θ = The angle which the balancing mass makes with m_1.

Since the magnitude of centrifugal forces are proportional to the product of each mass and its radius, therefore

$$m_1 \cdot r_1 = 200 \times 0.2 = 40 \text{ kg-m}$$

$$m_2 \cdot r_2 = 300 \times 0.15 = 45 \text{ kg-m}$$

$$m_3 \cdot r_3 = 240 \times 0.25 = 60 \text{ kg-m}$$

$$m_4 \cdot r_4 = 260 \times 0.3 = 78 \text{ kg-m}$$

The problem may, now, be solved either analytically or graphically. But we shall solve the problem by both the methods one by one.

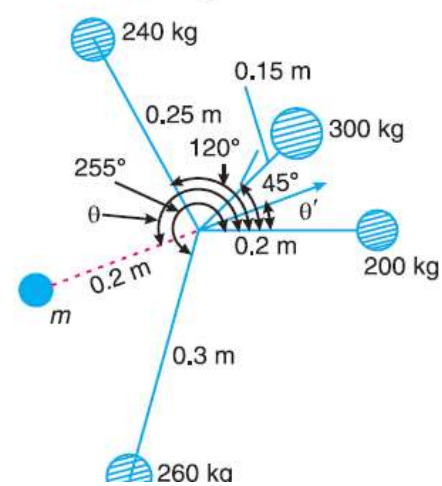


Fig 35

3.4.6. Balancing of Several Masses Rotating in Different Planes

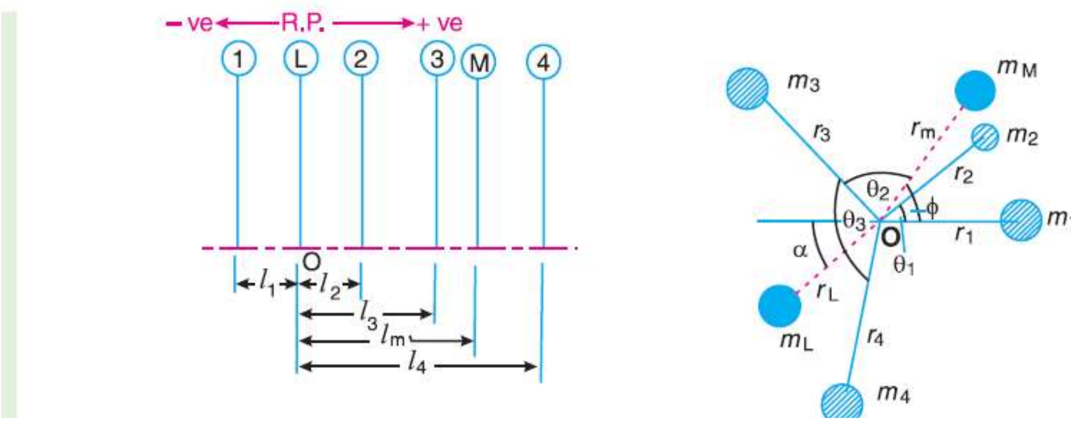
When several masses revolve in different planes, they may be transferred to a ***reference plane*** (briefly written as ***R.P.***), which may be defined as the plane passing through a point on the axis of rotation and perpendicular to it. The effect of transferring a revolving mass (in one plane) to a reference plane is to cause a force of magnitude equal to the centrifugal force of the revolving mass to act in the reference plane, together with a couple of magnitude equal to the product of the force and the distance between the plane of rotation and the reference plane. In order to have a complete balance of the several revolving masses in different planes, the following two conditions must be satisfied:

1. The forces in the reference plane must balance, *i.e.* the resultant force must be zero.

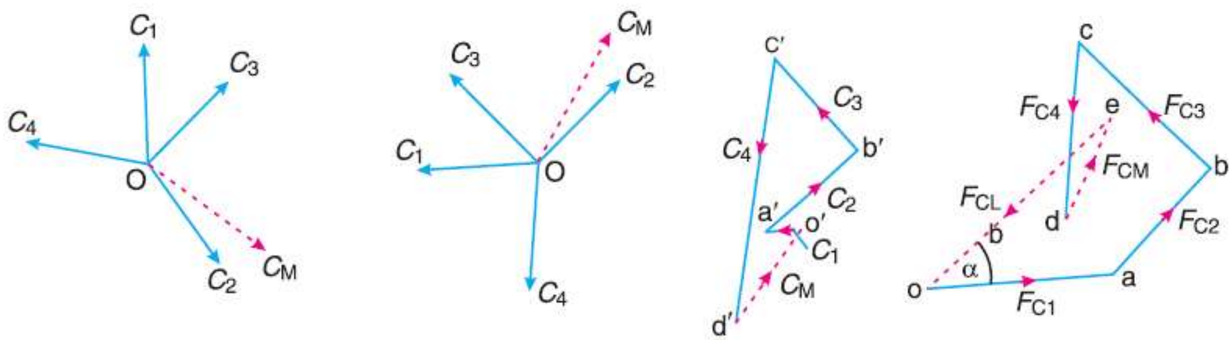
2. The couples about the reference plane must balance, *i.e*. the resultant couple must be zero. Let us now consider four masses *m*1, *m*2, *m*3 and *m*4 revolving in planes 1, 2, 3 and 4 respectively as shown in Fig. 21.7 (*a*). The relative angular positions of these masses are shown in the end view [Fig. 36 (*b*)].
The magnitude of the balancing masses m_L and m_M in planes *L* and *M* may be obtained as discussed below:
1. Take one of the planes, say *L* as the reference plane (*R.P*.). The distances of all the other planes to the left of the reference plane may be regarded as ***negative***, and those to the right as ***positive***.
2. Tabulate the data as shown in Table 3.1. The planes are tabulated in the same order in which they occur, reading from left to right.

Fig. 3.1

Plane (1)	*Mass (m)* (2)	*Radius(r)* (3)	*Cent.force* $\div \omega^2$ *(m.r)* (4)	*Distance from Plane L (l)* (5)	*Couple* $\div \omega^2$ *(m.r.l)* (6)
1	m_1	r_1	$m_1.r_1$	$-l_1$	$-m_1.r_1.l_1$
L(R.P.)	m_L	r_L	$m_L.r_L$	0	0
2	m_2	r_2	$m_2.r_2$	l_2	$m_2.r_2.l_2$
3	m_3	r_3	$m_3.r_3$	l_3	$m_3.r_3.l_3$
M	m_M	r_M	$m_M.r_M$	l_M	$m_M.r_M.l_M$
4	m_4	r_4	$m_4.r_4$	l_4	$m_4.r_4.l_4$



(*a*) Position of planes of the masses. (*b*) Angular position of the masses.



(*c*) Couple vector. (*d*) Couple vectors turned counter clockwise through a right angle. (*e*) Couple polygon. (*f*) Force polygon.

Fig. 36. Balancing of several masses rotating in different planes.

3. A couple may be represented by a vector drawn perpendicular to the plane of the couple. The couple C_1 introduced by transferring m_1 to the reference plane through O is propor tional to $m_1.r_1.l_1$ and acts in a plane through Om_1 and perpendicular to the paper. The vector representing this couple is drawn in the plane of the paper and perpendicular to Om_1 as shown by OC_1 in Fig.36 (*c*). Similarly, the vectors OC_2, OC_3 and $OC_4$are drawn perpendicular to Om_2, Om_3 and Om_4 respectively and in the plane of the paper.

4. The couple vectors as discussed above, are turned counter clockwise through a right angle for convenience of drawing as shown in Fig.36 (*d*).

We see that their relative positions remains unaffected. Now the vectors OC_2, OC_3 and OC_4 are parallel and in the same direction as Om_2, Om_3 and Om_4, while the vector OC_1 is parallel to Om_1 but in *opposite direction. Hence the ***couple vectors are drawn radially outwards for the masses on one side of the reference plane and radially inward for the masses on the other side of the reference plane.***

5. Now draw the couple polygon as shown in Fig.36 (*e*).

The vector $d'\,o'$ represents the balanced couple. Since the balanced couple C_M is proportional to $m_M . r_M . l_M$, therefore

$$c_M = m_M \,.\, r_M \,.\, l_M = \text{vector d' o' or } m_M = \frac{vector\ d'o'}{r_M \,.\, l_M}$$

From this expression, the value of the balancing mass m_M in the plane M may be obtained, and the angle of inclination ∅ of this mass may be measured from Fig. 36(*b*).

6. Now draw the force polygon as shown in Fig.36 (*f*). The vector *eo* (in the direction from *e* to *o*) represents the balanced force. Since the balanced force is proportional to $m_L . r_L$, therefore,

$$m_M \cdot r_M = \text{vector } e\,o \text{ or } m_L = \frac{\text{vector eo}}{r_L}$$

From this expression, the value of the balancing mass m_L in the plane L may be obtained and the angle of inclination α of this mass with the horizontal may be