MECHANICS OF MACHINE I (MCE 312)

Introduction

1.1 Mechanics

Mechanics is defined as a science that concerned with the motion of bodies under the action of forces, including the special case in which a body remains at rest. Of first concern in the problem of motion are the forces that bodies exert on one another. This leads to the study of such topics as gravitation, electricity, and magnetism, according to the nature of the forces involved. Given the forces, one can seek the manner in which bodies move under the action of forces; this is the subject matter of mechanics proper.

Historically, mechanics was among the first of the exact sciences to be developed. Its internal beauty as a mathematical discipline and its early remarkable success in accounting in quantitative detail for the motions of the Moon, the Earth, and other planetary bodies had enormous influence on philosophical thought and provided impetus for the systematic development of science into the 20th century.

Mechanics may be divided into three branches: statics, which deals with forces acting on and in a body at rest; kinematics, which describes the possible motions of a body or system of bodies; and kinetics, which attempts to explain or predict the motion that will occur in a given situation. Alternatively, mechanics may be divided according to the kind of system studied. The simplest mechanical system is the particle, defined as a body so small that its shape and internal structure are of no consequence in the given problem. More complicated is the motion of a system of two or more particles that exert forces on one another and possibly undergo forces exerted by bodies outside of the system.

The principles of mechanics have been applied to three general realms of phenomena. The motions of such celestial bodies as stars, planets, and satellites can be predicted with great accuracy thousands of years before they occur. (The theory of relativity predicts some deviations from the motion according to classical, or Newtonian, mechanics; however, these are so small as to be observable only with very accurate techniques, except in problems involving all or a large portion of the detectable universe.) As the second realm, ordinary objects on Earth down to microscopic size (moving at speeds much lower than that of light) are properly described by classical mechanics without significant corrections. The engineer who designs bridges or aircraft may use the Newtonian laws of classical mechanics with confidence, even though the forces may be very complicated, and the calculations lack the beautiful simplicity of celestial mechanics. The third realm of phenomena comprises the behaviour of matter and electromagnetic radiation on the atomic and subatomic scale. Although there were some limited early successes in describing the behaviour of atoms in terms of classical mechanics, these phenomena are properly treated in quantum mechanics.

Classical mechanics deals with the motion of bodies under the influence of forces or with the equilibrium of bodies when all forces are balanced. The subject may be thought of as the elaboration and application of basic postulates first enunciated by Isaac Newton in his

Philosophiae Naturalis Principia Mathematica (1687), commonly known as the *Principia*. These postulates, called Newton's laws of motion, are set forth below. They may be used to predict with great precision a wide variety of phenomena ranging from the motion of individual particles to the interactions of highly complex systems. A variety of these applications are discussed in this article.

In the framework of modern physics, classical mechanics can be understood to be an approximation arising out of the more profound laws of quantum mechanics and the theory of relativity. However, that view of the subject's place greatly undervalues its importance in forming the context, language, and intuition of modern science and scientists. Our present-day view of the world and man's place in it is firmly rooted in classical mechanics. Moreover, many ideas and results of classical mechanics survive and play an important part in the new physics.

The central concepts in classical mechanics are force, mass, and motion. Neither force nor mass is very clearly defined by Newton, and both have been the subject of much philosophical speculation since Newton. Both of them are best known by their effects. Mass is a measure of the tendency of a body to resist changes in its state of motion. Forces, on the other hand, accelerate bodies, which is to say, they change the state of motion of bodies to which they are applied. The interplay of these effects is the principal theme of classical mechanics.

Although Newton's laws focus attention on force and mass, three other quantities take on special importance because their total amount never changes. These three quantities are energy, (linear) momentum, and angular momentum. Any one of these can be shifted from one body or system of bodies to another. In addition, energy may change form while associated with a single system, appearing as kinetic energy, the energy of motion; potential energy, the energy of position; heat, or internal energy, associated with the random motions of the atoms or molecules composing any real body; or any combination of the three. Nevertheless, the total energy, momentum, and angular momentum in the universe never changes. This fact is expressed in physics by saying that energy, momentum, and angular momentum are conserved. These three conservation laws arise out of Newton's laws, but Newton himself did not express them. They had to be discovered later.

It is a remarkable fact that, although Newton's laws are no longer considered to be fundamental, nor even exactly correct, the three conservation laws derived from Newton's laws—the conservation of energy, momentum, and angular momentum—remain exactly true even in quantum mechanics and relativity. In fact, in modern physics, force is no longer a central concept, and mass is only one of a number of attributes of matter. Energy, momentum, and angular momentum, however, still firmly hold centre stage. The continuing importance of these ideas inherited from classical mechanics may help to explain why this subject retains such great importance in science today.

1.2 Mechanics of Machines

The subject Mechanics of Machines may be defined as that branch of Engineering science which deals with the study of relative motion between the various parts of a machine and forces which act on these parts due to constrained motion. The study of relative motion alone is referred to as Kinematics while that of forces acting on these parts is called dynamics.

Kinematics

Velocity: This is defined as the rate of change of displacement linear and angular of a body with respect to the time. Velocity is a vector quantity, to specify it completely the magnitude, direction and sense must be known.

Acceleration: The acceleration of a body is the rate of change of its velocity linear or angular with respect to time. A body accelerates if there is a change in either the magnitude, direction or sense of its velocity and can thus accelerates without change in speed, as in the case of a body moving in a circular path with uniform speed.

Displacement: Displacement is defined as the distance moved by a body with respect to a certain fixed point. The displacement may be along a straight or a curved path.

Equations of Uniformly accelerated Motion.

Let a body having linear motion accelerates uniformly from an initial velocity **u** to a final velocity **v** in time t; let the acceleration be **a** and the distance from the initial position be **s**.

Then

$$V = u + at$$

$$S = ut + \frac{1}{2}at^2 \quad ,$$

$$V^2 = u^2 + 2as$$

The corresponding equations for angular motion are:

$$\omega_2 = \omega_1 + \alpha t$$

$$\theta = \omega_1 t + \frac{1}{2}\alpha t^2 \quad ,$$

$$\omega_2{}^2 = \omega_1{}^2 + 2\alpha\theta$$

Where ω_2 *and* ω_1 are the initial and final angular velocities respectively, θ is the angle turned through in time t and $\alpha$is the angular acceleration.

Note: If a body is rotating at the rate of N rpm (revolution per minute), then its angular velocity,

$$\omega = \frac{2\pi N}{60} \text{ rad/ s}$$

Non-Uniform acceleration

If the non acceleration is a function of time , distance of velocity, it must be expressed in the form

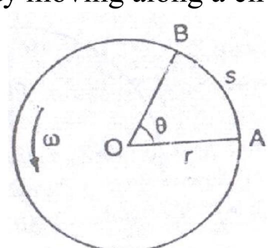
$$a = \frac{dv}{dt} = \frac{d}{dt}\frac{ds}{dt} = \frac{d^2s}{dt^2}\text{-+}\ldots\ldots\ldots\ldots \quad \textit{Note:}\ v = \frac{ds}{dt}$$

OR

$$a = \frac{dv}{dt} = \frac{ds}{dt} \times \frac{dv}{ds} = v \times \frac{dv}{ds}$$

Relation between Linear and Angular Quantities of Motion.

Consider a body moving along a circular path from A to B as shown in fig. 1.1 below,



Let r = Radius of the circular path

θ= Angular displacement in radians

S= Linear displacement

V = Linear velocity

ω = Angular velocity

a = Linear acceleration, and

α = Angular acceleration

Find the geometry of the figure, we know that

$$s = r.\theta \quad or \quad \theta = s/r$$

` Also, from linear velocity

$$V = \frac{ds}{dt} = \frac{d(r.\theta)}{dt} = r.\frac{d\theta}{dt} = r.\omega$$

and linear acceleration

$$a = \frac{dv}{dt} = \frac{d(r.\omega)}{dt} = r.\frac{d\omega}{dt} = r.\alpha$$

Note:

In the case of a wheel or cylinder which rolls without slip on a flat surface

WORKED EXAMPLES

Example 1

There motion of a particle is given by a= t^3-3t^2+5. Where a is the acceleration in m/s^2 and t is the time in seconds. The velocity of the particle at t=1 second is 6.25m/s and the displacement is 8.30metres. calculate the displacement and the velocity at t=2 seconds.

Given: $a = t^3 - 3t^2 + 5$

From $a = \frac{dv}{dt}$,

$$\frac{dv}{dt} = t^3 - 3t^2 + 5$$

or $dv = \left(t^3 - 3t^2 + 5\right)dt$

Integrating both sides

$$V = \frac{t^4}{4} - \frac{3t^3}{3} + 5t + c_1$$

$$= \frac{t^4}{4} - t^3 + 5t + c_1 \text{ ....}(i)$$

where c_1 is the first constant of integration, from the question, when t=1s, v=6.25m/s, therefore substituting these values of t and v in equation (i).

$$6.25 = 0.25 - 1.5 + c_1$$

$$= 4.25 + c_1$$

$$or\ c_1 = 6.25 - 4.25$$

$$= 2$$

Now, substituting the value of c_1 in equation (i)

$$V = \frac{t^4}{4} - t^3 + 5t + 2 \text{ ....}(ii)$$

When velocity at t = 2seconds

$$V = \frac{2^4}{4} - 2^3 + 5 \times 2 + 2$$

$$= 8m/s$$

When displacement at t=seconds

From, V=$\frac{ds}{dt}$, therefore equation (ii) may be written as

$$\frac{ds}{dt} = \frac{t^4}{4} - t^3 + 5t + 2$$

OR $\quad ds = \left(\frac{t^4}{4} - t^3 + 5t + 2\right)dt$

Integrating both sides

$$S = \frac{t^4}{20} - \frac{t^4}{4} + \frac{5t^2}{2} + 2t + c_2 \text{.....................}(iii)$$

Where ,

c_2 is the second constant of integration

When t=1s, s=8.30m, substituting these values in equation (iii),

$$8.30 = \frac{1}{20} - \frac{1}{4} + \frac{5}{2} + 2 + c_2$$

$$= 4.3 + c_2$$

$$c_2 = 8.3 - 4.3$$

$$= 4$$

Substituting the values of c_2 in equation (iii)

$$s = \frac{t^5}{20} - \frac{t^4}{4} + \frac{5t^2}{2} + 2t + 4.$$

Substituting the value of t = 2 seconds in this equation, we have

$$S = \frac{2^5}{20} - \frac{2^4}{4} + \frac{5 \times 2^2}{2} + 2 \times 2 + 4$$

= 15.6m.

Example 2.

A wheel accelerates uniformly from rest to 2000 rpm in 20 seconds. What is its angular acceleration? How many revolutions does the wheel make in attaining the speed of 2000 rpm.

Solution

Given: $N_o = 0$, or $\omega = 0$; N = 2000 r.p.m or $\omega = 2\prod \times \frac{2000}{60} = 209.5 rad/s$,

t = 20 seconds

Let α = Angular acceleration in rad/s^2

From $\omega = \omega_o + \alpha t \; or \; 209.5 = 0 + \alpha \times 20$

$$\alpha = \frac{209.5}{20}$$
$$= 10.475 rad/s^2$$
$$\theta = \frac{(\omega_o + \omega)t}{2} = \frac{(0 + 209.5) \times 20}{2} = 2095 \, rad$$

Since the angular distance moved by the wheel during one revolution is 2π radians, therefore the number of revolutions made by the wheel,

$$n = \theta / 2\pi = \frac{2095}{2\pi} = 333.4$$

Assignment

(1) A horizontal bar 1.5metres long nad of small cross-section rotates about vertical axis through one end. It accelerates uniformly from 1200r.p.m to 1500 r.p.m in an interval of 5 seconds. What is the linear velocity at the beginning and end of the interval? What are the normal and tangential components of the acceleration of the mid-point of the bar 5 seconds after acceleration begins?

(2) The displacement of a point is given by $S = 2t^3 + t^2 + 6$, where S is in metres and t in seconds. Determine the replacement of the point when the velocity changes from 8.4m/s to 18m/s. Find also the acceleration at the instant when the velocity of the particle is 30m/s.

DYNAMICS

1.2 Mass, Force, Weight and Momentum

- **Mass**: It is the amount of matter contained in a given body, and does not vary with the change in its position on the earth's surface. The mass of a body is measured by direct comparison with a standard mass by using a lever balance.

- **Weight:** It is the amount of pull, which the earth exerts upon a given body. Since the pull varies with distance of the body from the centre of the earth, therefore the weight of the body will vary with its position on the earth's surface (say latitude and elevation). It is thus obvious, that the weight is a force.

- **Force:** It is an important factor in the field of engineering science, which may be defined as an agent, which produces or tend to produce, destroy or tend to destroy motion.
- **Momentum:** The momentum of a body is the product of its mass and velocity. Mathematically.

Momentum = mass x velocity

Let M = mass of the body

u = Initial velocity of the body

v = final velocity of the body

a = constant acceleration, and

t = Time required (in seconds) to change the velocity from u to v.

now, initial momentum = M.u

and final momentum = m.v

$\therefore$ change of momentum= m.v – m.u

and rate of change of Momentum = $\frac{m.v - m.u}{t} = \frac{m(v-u)}{t} = m.a$

Newton's Law of Motion

1. Everybody continues in its state of rest or of uniform motion in a straight line, unless acted upon by some external force.
2. The rate of change of momentum is directly proportional to the impressed force and takes place in the same direction in which the forces acts.
3. To every action, there is always an equal and opposite reaction.

From the second law,

Force α rate of change of momentum

α mass x rate of change of velocity

i.e F = kma

where k is a constant

The unit of the quantities are chose so as to make the value of k unity.

i.e. F = ma.

Which is the force required to give a mass of 1kg an acceleration of $1m/s^2$

- **Impulse:** The impulse of a constant force (f) acting for a time t is the product ft. if during this time, the velocity change from u to v, then,

$$f = ma = \frac{m(v-u)}{t}$$

$$or$$

$$ft = m(v-u)$$

$\therefore$ impulse of force = change of momentum

Relative Velocity

If two bodies A and B are moving with velocities V_a and V_b respectively as in fig 1.2 ,then the relative velocity of one to the other is the vector difference of v_a and v_b,

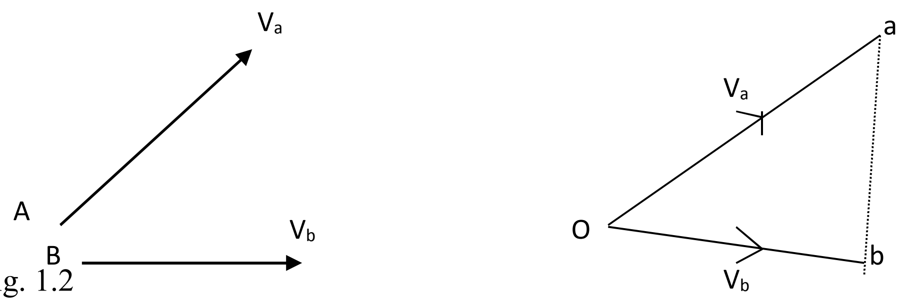


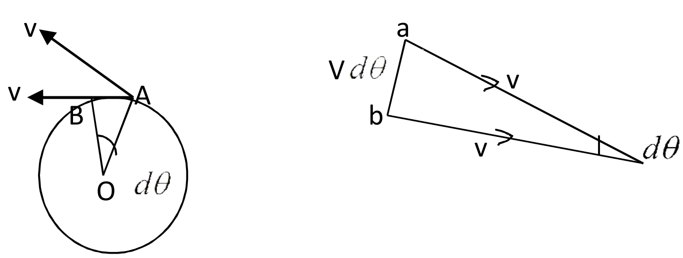
Fig. 1.2

i.e. if vectors oa and ob, representing Va and Vb in magnitude, direction and sense, are drawn from the same point o, then ab represents the velocity of B relative to A and ba the velocity of A relative to B.

If Oa and Ob represent the velocities of the same body at different times, then ab represents the change in velocity.

Centripetal acceleration and centrifugal force

Consider a body of mass m moving in a circular path of radius r with constant speed V, fig. 1.3. If it moves from A to B in time dt and the angle AOB is dθ



then, from the relative velocity diagram, the change of velocity is represented by ab

Thus, change of velocity $= Vd\theta$

$$\therefore \ acceleration = V\frac{d\theta}{dt}$$

i.e. $a = v\omega$

where ω is the angular velocity of OA

But $V = \omega r$ ....*recall*

$$\therefore \quad a = \omega^2 r \ \ or \ \ {v^2}/{r}$$

This linear acceleration derived directed towards the centre of rotation o is called **Centripetal acceleration**. The out-of-balance force causing this acceleration is called **Centripetal force**. The radially inward, or centripetal force required to produce this acceleration is given by

$$F = ma = m\omega^2 r$$

Or $= {mv^2}/{r}$

If a body rotates at the end of an arm, this force is provided by the tension in the arm. The reaction to this force acts at the centre of rotation and is called the centripetal force.

In many engineering problems, there is often, though not always, an equal and opposite reaction to centripetal force called **centrifugal force**. This force never acts directly on the body moving in a circle but exists only as a reaction to the centripetal force. A common concept of centrifugal force in engineering problems is to regard it as the radially outward force which must be applied to a body to convert the dynamical condition to the equivalent static condition; this is known as d'Alembert's principle.

Mass Moment of Inertia

If the mass of every particle of a body is multiplied by the square of its perpendicular distance from a fixed line, then the sum of these quantities (for the whole body) is known as mass moment of inertia of the body. It is denoted by I.

Consider a body of total mass compose of small particles of masses m_1, m_2, m_3, m_4 etc. If k_1, k_2, k_3, k_4 are the distances of these masses from a fixed line, as shown in fig 1.4, then the mass moment of Inertia of the whole body is given by

$$I = m_1(k_1)^2 + m_2(k_2)^2 + m_3(k_3)^2 + m_4(k_4)^2 + ...$$

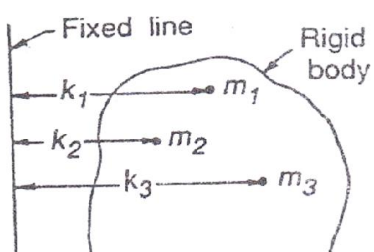


Fig. 1.4

If the total mass of body may be assumed to concentrate at one point (known as centre of mass or centre of gravity), at a distance k from the given axis, such that

$$m.k^2 = m_1(k_1)^2 + m_2(k_2)^2 + m_3(k_3)^2 + m_4(k_4)^2 + ...$$

$$then \quad I = m.k^2$$

The distance K is called the **radius of Gyration.** This is defined as the distance from a given reference where the whole mass of body is assumed to be concentrated to give the same value of I.

Angular Momentum or Moment of Momentum

Consider a body of total mass m rotating with an angular velocity of ω rad/s, about the fixed axis O as shown in fig. 1.5. Since the body is composed of numerous small particles, therefore let us take one of these small particles having a mass dm and at a distance r from the axis of rotation. Let v be its linear velocity acting tangentially at any instant.

Recall, momentum is the product of mass and velocity, therefore momentum of mass dm

$$= dm \times v = dm \times \omega \times r. ....(\therefore V = \omega.r)$$

and

moment of momentum of mass dm about O

$$= dm \times \omega \times r \times r = dm\,x\,r^2 \times \omega = I_m \text{ x } \omega$$

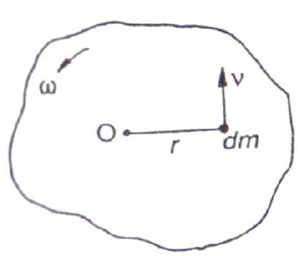


Fig. 1.5

Where I_m = Mass moment of Inertia of mass dm about O = dm x r^2

$\therefore$ Moment of momentum or angular momentum of the whole body about 0

$$= \int I_m \omega = I.\omega$$

Where $\int I_m$ = Mass moment of Inertia of the whole body about O

Thus, the angular momentum or the moment of momentum is the product of mass moment of inertia (I) and the angular velocity (ω) of the body.

Torque

It may be defined as the product of force and the perpendicular distance of its line of action from the given point or axis.

The Newton's second Law of motion, when applied to rotating bodies, state that the torque is directly proportional to the rate of angular momentum. Mathematically

$$Torque,\ T \times \frac{d(I.\omega)}{dt}$$

$$T = I\ \frac{d\omega}{dt} = I.\alpha \ldots\ldots\ldots\ldots\ldots\ldots \left(\therefore \frac{d\omega}{dt} = \alpha \right)$$

Example 1

The flywheel of a steam engine has a radius of gyration of 1m and mass 2500 kg. The starting torque of the steam engine is 1500 Nm and may be assumed constant. Determine:

(i) angular acceleration of the flywheel

(ii) kinetic energy of the flywheel after 10 seconds from the start

Solution:

Given: K = 1m; m = 2500kg, T=1500Nm

Let α = Angular acceleration of the flywheel

From mass moment of inertia I = m.k^2

= 2500 x I^2

$= 2500 kgm^2$

We know that torque (T) = I. α

$$1500 = 2500 \text{ x } \alpha$$

Or

$$\alpha = \frac{1500}{2500} = 0.6 rad/s^2$$

(ii) We know that $\omega_2 = \omega_1 + \alpha .t$

$= 0+0.6 \text{ x } 10$

$= 6$ rad/s

$\therefore$ Kinetic energy of the flywheel,

$$E = \frac{1}{2} I(\omega_2)^2$$

$$= \frac{1}{2} \times 2500 \times 6^2$$

$$= 4500 J = 45 kJ$$

Example 2

A haulage rope winds on a drum of radius 500 mm, the free and being attached to a truck. The truck has a mass of 500 kg and is initially at rest. The drum is equivalent to a mass of 1250 kg with radius of gyration 450mm. The rim speed of the drum is 0.75m/s before the rope tightens. By considering the change in linear momentum of the truck and in the angular momentum of the drum, find the speed of the truck when the motion becomes steady. Find also the energy lost to the system.

Solution:

Given: r = 500mm = 0.5m, m_1 = 500kg, m_2 = 1250kg: k=450mm = 0.45m u=0.75m/s.

From,

$$I_2 = m_2 k^2 = 1250(0.45)^2$$

$$= 253 kgm^2$$

Speed of the truck

From,

F = m_1,v = 500v Ns

and moment of impulse = change in angular momentum of drum

$$f \times r = I_2(\omega_2 \omega_1) = I_2\left(\frac{u - v}{r}\right)$$

$$500v \times 0.5 = 253\left(\frac{0.75 - u}{0.5}\right)$$

$$250v + 506v = 380$$

$$V = 380/706$$

$$= 0.502 m/s$$

Energy lost to the system

From,

Energy lost to the system = Loss in K.E of drum – Gain in K.E of truck

$$= \frac{1}{2} \times I_2 \left[(\omega_2)^2 - (\omega_1)^2\right] - \frac{1}{2} \times m_1 v^2$$

$$= \frac{1}{2} \times I_2 \left[\frac{u^2 - v^2}{r^2}\right] - \frac{1}{2} m_1 v^2$$

$$= \frac{1}{2} \times 253 \left[\frac{(0.75)^2 - (0.502)^2}{(0.5)^2}\right] - \frac{1}{2} \times 500(0.5002)^2 Nm$$

$$= 94 Nm$$

Work

Whenever a force acts on a body and the body undergoes a displacement in the direction of the force, then work is said to be done. For example, if a force (F) acting on a body causes a displacement S of the body in the direction of the force, then

Work done= force x Displacement

= F x S

If the force varies linearly from zero to a maximum value of F, then

Work done = $\frac{O + F}{2} \times S = \frac{1}{2} \times F \times S$

It may happen that the line of action of the force is at an angle to the direction of motion of the body. For example, let a uniform force of 80N act on a body at 30° to the horizontal and let the body move a horizontal distance OA = 4m. the work done by the force is determined by the distance OC = 4 cos 30°, moved by the force along its line of action. Thus:

Work done = 80 x Oc

$= 80 \times 4Cos \cos 30^o$

= 277 J

Work done in particular cases

- ***Force of Gravity***

The force required to lift a body of mass m through a height h is equal to the weight, mg. Therefore the work done in overcoming the force of gravity is **mgh**.

For example, when an aircraft climbs at an angle θ to the horizontal and travels a distance S along the line of flight, then the centre of gravity of the plane is raised through a height **S Sin** θ, and the work done is **mg x S sin** θ.

Alternatively, the resolved part of the weight along the line of flight is **mg sin** θ and the work done against this force is S distance is **mg sin** θ **x s**, as before.

- **Resisting force**

If a body travels a distance S against a steady resisting force R,

then,

Work done against resistance = resistance x distance moved

= **Rs**

- **Accelerating Body**

Consider a body of mass m accelerated from rest with uniform acceleration a over a distance s, when there is no resistance to motion. The accelerating force F = ma, and the work done by the accelerating force is **Fx s = ma x s**

- **Body moving on an incline**

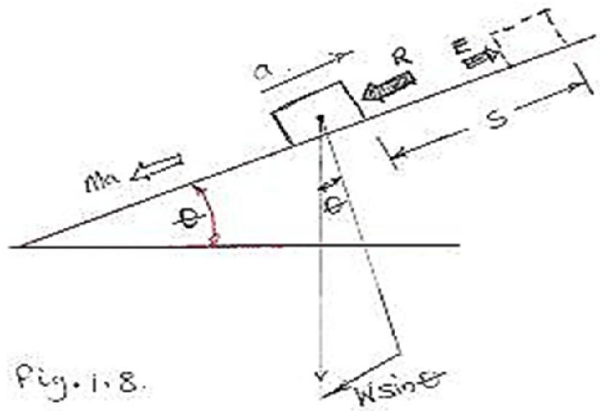
Fig. 1.6 shows the free-body diagram for the particular case of a body of mass m being accelerated up a gradient by an effort E against a constant resistance R.. The work done by the effort is E x S, and this is made up of the work done against the component of the weight acting down the slope, the resistance and the inertia force.

Thus,

Work done = E x S

= (W sin θ+R +ma) s.

Fig. 1.6



- **Work done by a torque**

Consider an arm OA of length r, rotating about an axis O, due to the action of a constant force F applied tangentially at A, fig. 1.7. The applied toque about O is Fr = T. If the arm turns through an angle θ then, the force F moves a distance rθ along the arc. Hence, The work done by the force is F x rθ or Tθ, i.e. the work done by a toque is the product of the torque and the angle turned through.

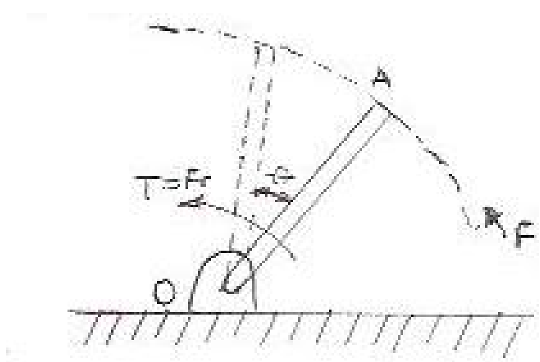


Fig. 1.7

Thus,

Work done by constant Torque T = Tθ

If the torque is applied gradually so that it varies linearly from zero to a maximum value T, then the average torque is ½ T and hence,

Work done by a gradually applied torque = ½Tθ

Energy

Energy may be defined as capacity to do work. The energy exists in many forms e.g. mechanical, electrical, chemical, heat, light etc. But we are mainly concerned with mechanical energy.

The mechanical energy is equal to the work done on a body in altering either its position, velocity or shape. Three types of mechanical energies are important from the subject point of view.

1. **Potential Energy (P.E)**: This is the energy possessed by a body due to its position and is equal to the work done in raising it from some datum level. Thus the P.E. of a body of mass m at a height h above datum level is **mgh**.
2. **Kinetic Energy (K.E)**: The Kinetic energy of a body is the energy it possesses due to its mass and velocity of the motion. If a body of mass m attains a velocity V from rest in time t under the influence of a force F and moves a distance S, then.

   Work done by F = F x S

   = ma x s

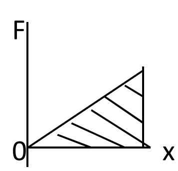
   ∴ Kinetic Energy of the body or Kinetic Energy of translation,

   $$K.E = mas = m \times a \frac{v^2}{2a}$$

   $$= \frac{1}{2}mv^2$$
3. **Strain Energy:** The strain energy of an elastic body is the energy stored when the body is deformed. In the simple case of a body which is strained from its natural position, the relation between the deformation x and the straining force F is a straight line passing through the origin, Fig. 1.8 and the work done is represented by the area of the triangle. Thus

Work done = strain energy = ½ Fx

Fig. 1.8.



$$Work\ done = \frac{1}{2}Sx^2$$

Where,

S = Stiffness (i.e. force per unit deformation)

In case of a torsional spring of stiffness q (force per unit angular deformation) when twisted through an angle θ radians, then

$$Strain\ energy = Work\ done = \frac{1}{2}q\theta^2$$

Principle of Conservation of Energy

Its state that “the energy can neither be created nor destroyed, though it can be transformed from one form into any of the forms, in which the energy can exist”.

Note: A loss of energy in any one form is always accompanied by an equivalent increase in another form. When work is done on a rigid body, the work is converted into kinetic or potential energy or is used in overcoming friction. If the body is elastic, some of the work will also` be stored as strain energy.

The following examples will show how this occurs.

- ***Falling body***: A falling body loses potential energy but gains a corresponding amount of kinetic energy.
- ***Mass spring***: A mass vibrating at the end of a spring loses kinetic energy in stretching the spring but the spring then possesses potential or strain energy. When the motion is reversed and the spring is acting on the mass. Its strain energy is transferred to the mass as kinetic energy of motion.
- ***Lost energy when friction is involved***: A body moving along a rough surface loses kinetic energy corresponding to the work done against the friction forces. This work is generally ‘lost’ for mechanical purposes but reappears as heat energy in the body and surface. Thus the total energy of the system, body and surface, is conserved.

The work done in overstretching a spring so that it is permanently deformed is again lost for mechanical purposes and converted into heat.

- **Collision of bodies:** when two perfectly elastic bodies collide, the work done in elastic deformation is recovered as they rebound. For example, when two steel balls bounce together, there is compression of the steel on impact but as they move apart, the steel recovers its shape and in doing so restores kinetic energy.

Principle of conservation of Linear Momentum

It states “the total momentum of a system of masses (i.e. moving bodies0 in any one direction remains constant, unless acted by an external force in that direction. “This principle is applied to problems on impact, i.e. collision of two bodies. In other words, if two bodies of masses m_1 and m_2 with linear velocities v_1 and v_2 are moving in the same straight line, and they collide and begin to move together with a common velocity v, then

Momentum before impact = momentum after impact

i.e. $$m_1 v_1 \pm m_2 v_2 = (m_1 + m_2)v$$

Note:

1. The positive sign is used when the two bodies move in the same direction after collision. The negative is used when they move in the opposite direction after collision.
2. Consider two rotating mass moment of inertia I_1 and I_2 are initially apart from each other and are made to engage as in the case of a clutch. If they reach a common angular velocity ω, after slipping has ceased, then

$$I_1 \omega_1 \pm I_2 \ \omega_2 = (I_1 + I_2)\omega$$

The $\pm$ sign depends upon the direction of rotation.

Energy Lost by Friction Clutch during Engagement

Consider two collinear shafts A and B connected by a friction clutch (plate or disc clutch)

Let I_A and I_B = mass moment of inertias of the rotors attached to shafts A and B respectively

ω_A and ω_B = Angular speeds of shaft A and B before engagement of clutch, and

ω = common angular speed of shafts A and B after engagement of clutch.

By the principle of conservation of momentum,

$$I_A \omega_A + I_B \ \omega_B = (I_A + I_B)\omega$$

$$\omega = \frac{I_A \ \omega_A + I_B . \omega_B}{I_A \ \omega_B} \qquad ....(i)$$

Total Kinetic energy of the system before engagement

$$E_1 = \frac{1}{2} I_A \; (\omega_A)^2 + \frac{1}{2} I_B \; (\omega_B)^2 = \frac{I_A(\omega_A)2 + I_B(\omega_B)^2}{2}$$

Kinetic energy of the system after engagement

$$E_2 = \frac{1}{2}(I_A + I_B)\omega^2 = \frac{1}{2}(I_A + I_B)\left[\frac{I_A\omega_A + I_B\omega_B}{I_A + I_B}\right]^2$$

$$= \frac{(I_A\omega_A + I_B\omega_B)^2}{2(I_A + I_B)} \; ....(ii)$$

$\therefore$ Loss of kinetic energy during engagement,

$$E = E_1 - E_2 = \frac{I_A(\omega_A)^2 + I_B(\omega_B)^2}{2} - \frac{(I_A\omega_A + I_B.\omega_B)^2}{2(I_A + I_B)}$$

$$= \frac{I_A \, .I_B(\omega_A - \omega_B)^2}{2(I_A + I_B)}$$

Note: (1) If the rotor attached to shaft B is at rest, then $\omega_B = 0$.

$$\therefore \omega = \frac{I_A \, .\omega_A}{I_A + I_B} \; ....(iii)$$

$$and \;\; loss \;\; of \;\; kinetic \;\; energy \;\; E = \frac{I_A + I_B(\omega_A)^2}{2(I_A + I_B)} \; ....(iv)$$

(2) If I_B is very small as compared to I_A and the rotor B is at rest, then,

$$\omega = \frac{I_A + \omega_A}{I_A + I_B} = \omega_A \; ....(Negleting \;\; I_B)$$

And from equation (iii) and (iv)

$$E = \frac{1}{2} I_B .\omega.\omega_A = \frac{1}{2} I_B .\omega^2$$

= Energy given to rotor B.

Example 1.

A force exerted on a body of mass 10kg travelling with an initial velocity 15m/s varies with time according to the relationship $F = 250 + 2t^2$. Calculate the total impulse imparted to the body in the first 3 seconds, and the final velocity of the body.

Solution

Total impulse = $\int_{t_1}^{t_2} Fdt$

$$\int_0^3 \left(250 + 2t^2\right) dt$$
$$= \left[250t + \frac{2t^3}{3}\right]_0^3$$
$$= [750 + 18] - [0]$$
$$= 768\ Nm$$

But total impulse = change in momentum of body

$$768 = m\ (v_2 - v_1)$$
$$= 10(v_2 - 15)$$
$$V_2 = \frac{768}{10} + 15$$
$$= 91.8 m/s$$

Example 2

A truck of mass 1600kg starts at rest and run down an incline of $\frac{1}{20}$. Calculate its velocity after it has travelled 30m, if the total frictional resistances to motion are 250N.

ASolution

Potential energy lost = Kinetic energy gained + work done against friction

or

mgh = ½ mv^2 + Fs.

$$\therefore 1600 \times g \times \frac{30}{20} = \frac{1}{2} \times 1600v^2 + 250 \times 30$$
$$2400 \times g = 800v^2 + 7500$$
$$v^2 = \frac{2400g - 7500}{800}$$
$$= \frac{23544 - 7500}{800}$$
$$= \frac{16044}{800}$$
$$= 20.06$$
$$v = 4.48 m/s$$

Example 3

The two buffers at one end of a truck each require a force of 0.7MN/m of compression and engage with similar buffers on a truck which it overtakes on a straight horizontal track.. The truck has a mass of 10 tonnes and its initial speed is 1.8m/s, while the second truck has mass of 15 tonnes with initial speed 0.6m/s, in the same direction.

Find (a) the common speed when moving together during impact

(b) the kinetic energy lost to the system

(c) the compression of each buffer to store the kinetic energy lost,

(d) the velocity of each truck on separation if only half of the energy offered in the springs is returned.

Solution

Given: $S = 0.7mN/m = 0.7\ x\ 10^6 N/m;\ m_1 = 10t = 10x10^3 kg$

$V_1 = 1.8m/s; m_2 = 15tonnes = 15\,x\,103kg,\ v_2 = 0.6m/s$

(a) Common velocity when moving together during impact

Let v = common velocity

We know that momentum before impact = momentum after impact

i.e. $m_1v_1 + m_2v_2\ (m_1 + m_2)v$

$$10\ x\ 10^3\ x1.8 + 15x\ 10^3\ x\ 0.6 = (10x\ 10^3\ + 15\ x\ 10^3)v$$

$$27\ x\ 10^3\ = 25\ x\ 10^3 v$$

$$v = \frac{27\ x\ 10^3}{25\ x\ 10^3} = 1.08 m/s$$

(b) Kinetic Energy lost to the system

Since the kinetic energy lost to the system is the kinetic energy before impact **minus** the kinetic energy after impact, therefore

Kinetic energy lost to the system

$$= \left(\frac{1}{2}m_1v_1^{\ 2} + \frac{1}{2}m_2v_2\right) - \frac{1}{2}(m_1 + m_2)v^2$$

$$= \left[\frac{1}{2} \times\ 10 \times 10^3 (1.8)^2 + \frac{1}{2} 15 \times 10^3\ (0.6)^2\ \right]$$

$$\frac{1}{2}\left(10 \times 10^3 + 15 \times 10^3\right)(1.08)^2$$
$$= 4.35 \ x \ 10^3 \ \ Nm = 4.35KNm \ (4350J \ or \ 4.35kJ)$$

(c) Compression of each buffer spring to store kinetic energy lost

Let x = compression of each buffer spring in metres

and S = force required by each buffer spring or stiffness of each spring = 0.7MN/m = 0.7 x 10^6m

since the strain energy stored in the springs (four in number) is equal to kinetic energy lost in impact, therefore,

$$4 \times \frac{1}{2} \ S.x^2 = 4.35 \times 10^3$$
$$4 \times \frac{1}{2} \ x \ 0.7 \times 10^6 \ x^2 = 4.35 \times 10^3$$
$$1.4 \times 10^6 \ x^2 = 4.35 \times 10^3$$
$$x^2 = \frac{4.35 \times 10^3}{1.4 \times 10^6}$$
$$= 3.11 \times 10^{-3}$$
$$x = 0.056m$$
$$= 56mm$$

(d) Final Kinetic energy after separation

= Kinetic energy at common velocity+ $\frac{1}{2}$ energy stored in springs

$$= \frac{1}{2} m_1 (v_3)^2 + \frac{1}{2} m_2 (v_4)^2 = \frac{1}{2}(m_1 + m_2)v^2 + \frac{1}{2}\left(4 \times \frac{1}{2} s.x^2\right)$$
$$= \frac{1}{2} \ x \ 10 \ x \ 10^3 + \frac{1}{2} \ x \ 15 \ x \ 10^3 \ (v_3)^2 = \frac{1}{2}\left(10 \ x \ 10^3 + 15x10^3\right)(1.08)^2 + \frac{1}{2}\left(4.35 \ x \ 10^3\right)$$
$$= 10 \ (v_3)^2 + 15 \ (v_4)^2 = 33.51 \text{..}(i)$$

We know that initial momentum and final momentum must be equal i.e.

$$m_1 v_3 + m_2 \ v_4 = (m_1 + m_2)v$$
$$10 \times 10^3 \times v_3 + 15 \ x \ 10^3 \times v_4 = \left(10 \times 10^3 + 15 \times 10_3\right)1.08$$
$$10v_3 + 15v_4 = 27$$

From equation (i) and (ii)

$$v_3 = 0.6m/s \ \ and \ v_3 = 1.4m/s$$

Simple Harmonic Motion

A particular is said to move with simple harmonic motion (SHM) when its acceleration is directly proportional to its displacement from a fixed point in its path, and always directed towards that point.

A perfect examples of SHM is the oscillation of a mass spring system. A spring of natural length l carries a mass of m kg. From its equilibrium position the mass is given a further downward displacement a and released. The mass moves with SHM since the restoring force in the spring, and hence the acceleration, is always directed towards the equilibrium position.

Simple harmonic motion can also be represented by the movement of a point which is the projection on a diameter of a second point moving in a circle with constant angular velocity.

Velocity and Acceleration of a Particle Moving with Simple Harmonic Motion

Consider a particle, moving round the circumference of a circle of radius r, with a uniform angular velocity ω rad/s as shown in fig.1.9. Let p be any position r a point of the particle after t seconds and θ be the angle turned by the particle in t seconds. We know that

$\theta = \omega.t.$

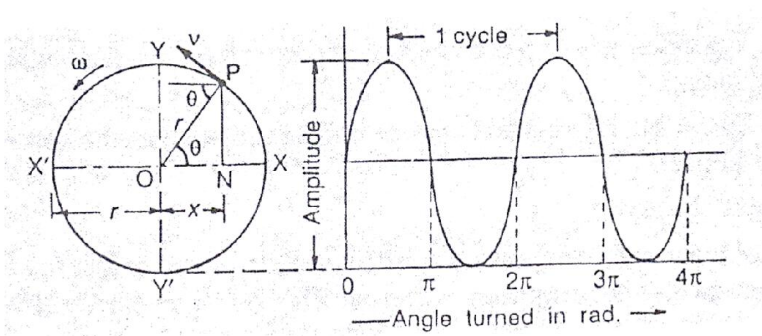


Fig. 1.9 Velocity and Acceleration of a particle

If N is the projection of p on the diameter xx^1, then displacement of N from its mean position 0 is

X= r.cos θ= r. cos ω.t. ……………………………….(i)

The velocity of n is the component of the velocity of p parallel to xx^{1}. i,e

V_N = v sin θ = ω.r sin θ = $\omega\sqrt{r^2 - x^2}$ …………………(ii)

……..……… $(\because v = \omega r, and\ r\sin\theta = Np = \sqrt{r^2 - x^2})$

A little consideration will show that velocity is maximum, when x = 0, i.e, when N passes through O; i.e its mean position.

V_{max} = ω.r,

We know that the acceleration of P is the centripetal acceleration whose magnitude is ω^2 r. The acceleration of N is the component of the acceleration of P parallel to xx^1 and is directed towards the centre 0., i.e.

a_N = ω^2.r cos θ = $\omega^2 \cdot x$ …………….. $(\because x = r\cos\theta)$ ……………………………(iii)

The acceleration is maximum when x = r ie. When P is at x or x^1

$a_{max} = \omega^2 r$

Different Equation of Simple Harmonic Motion

From,

x = r cos θ = r. Cos ωt …………………… (i)

Differentiating equation (i) is, we have velocity of N,

$\frac{dx}{dx}$ = V_N = -rω sin ωt………………………. (ii)

Again differentiating equation (ii), we have velocity of N,

$$\frac{d^2x}{dx^2} = V_N = a_N = -r.\omega.\omega \cos \omega t$$

$$= -\omega^2.r \cos \omega t \ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

$$= -\omega^2 x$$

Or.

$$\frac{d^2x}{dx^2} + \omega^2 x \quad = 0$$

This is the standard differential equation for simple harmonic motion of a particle. The solution of which is

$$x = A \cos \omega t + B \sin \omega t \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)}$$

Where A and B are constants to be determined by the initial conditions of the motion.

In fig. 1.9, when t = 0, $x = r$ i.e when point P and N lies at x, we have from equation (iv)

A = r.

Differentiating equation (iv)

$$\frac{dx}{dt} = -A.\omega \sin \omega t + B.\omega \cos \omega t. \ldots\ldots\ldots\ldots\ldots\ldots\text{(v)}$$

When t = 0, $\frac{dx}{dt} = 0$, therefore, from the above equation, B = 0.

Now the equation (iv) becomes

$$x = r \cos \omega t \ldots\ldots\ldots\ldots\ldots\ldots\text{(vi).}$$

The equations (ii) and (iii) may be written as in equation (vii) and (viii).

By differentiating equation (vi) twice.

$$\frac{dx}{dt} = V_N = -r.\omega \sin \omega t = -\omega r \cos(\omega t + \pi/2) \text{..........................}(vii).$$

And

$$\frac{d^2x}{dt^2} = a_N = -\omega^2.r \cos \omega t = \omega^2 r \cos(\omega t + \pi) \text{..................(viii)}$$

Example 1.

The piston of a steam engine moves with simple harmonic motion. The crank rotates at 120rpm with a stroke of 2 metres. Find the velocity and acceleration of the piston, when it is at a distance of 0.75m from the centre.

Solution

Given: N = 120rpm, r = 1m, $x = 0.75m$.

From $\omega = \frac{2\pi N}{60} = \frac{2\pi \times 120}{60}$

$= 4\pi \, rad/s$.

Velocity of the piston.

$V = w\sqrt{r^2 - x^2}$

$= 4\pi\sqrt{1 - 0.75^2}$

= 8.31m/s

Acceleration of the Piston

$a = \omega^2.x$

$= (4\pi)^2 \times 0.75$

$= 118.46 m/s^2$

Example 2

A body performs s.h.m in a straight line. Its velocity is 4m/s when the displacement is 50mm, and 3m/s when the displacement is 100mm, the displacement being measured from the mid position. Calculate the frequency and amplitude of the motion. What is the acceleration. What is the acceleration when the displacement is 75mm?

From,

$$V = \omega\sqrt{r^2 - x^2}$$

When $x = 0{,}05m \ and \ v = 4m/s$

$$4 = 10\sqrt{(r^2 - 0.05^2)} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(i).$$

When $x = 0.1m \ and \ v = 3m/s$.

$$3 = w\sqrt{(r^2 - 0.1^2)} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(ii)$$

Divide equation (i) by equation (ii)

$$\frac{4}{3} = \frac{\sqrt{r^2 - 0.05^2}}{\sqrt{r^2 - 0.1^2}}$$

Squaring both sides.

$$\frac{16}{9} = \frac{r^2 - 0.05^2}{r^2 - 0.1^2}$$

$$7r^2 = 0.1375.$$

Hence r = 0.140m.

= 140mm.

To find ω, from equation (i).

$$4 = \omega\sqrt{(0.140^2 - 0.0025^2)}\ .$$

Squaring both sides.

$$16 = \omega^2 (0.140^2 - 0.0023^2).$$

$$\omega^2 = \frac{16}{0.0196}$$
$$\omega = 28.6 rad/s$$

Thus, frequency $n = \dfrac{\omega}{2\pi}$

$$= \frac{28.6}{2\pi}$$

= 4.56Hz

The acceleration when $x = 0.075m$ is given by

$$a = \omega^2 x$$
$$= 28.6\ x\ 0.075$$
$$= 61.3 m/s^2$$

Assignment

1. A mass of 300kg is allowed to fall vertically through 1 metre on to the top of a pile of mass 500kg. Assume that the falling mass and pile remain in contact after impact and that the pile is moved 150mm at each blow. Find, allowing for the action of gravity after impact.

a. The energy lost in the blow

b. The average resistance against the pile

2. The force f on a body of mass 50kg varies with the distance of the body from a fixed point according to the law

$$F = 202.5 - 25 \text{ KN.}$$

Calculate the velocity of the body after it has moved a distance of 200m from rest, and the work done by the force in achieving this velocity.

3. A drop hammer is allowed to fall from rest through a height of 6m on to a forging. Find the downward velocity of the hammer when is strikes the forging. If the mass of the hammer is 500kg, what is the work done by the forging and base plate in bringing the hammer to rest in a distance of 500mm?

4. A wagon of mass 12000 kg traveling at 16 km/h strikes s pair of parallel spring-loaded stops. If the stiffness of each spring is 600 kN/m, calculate the maximum compression in bringing the wagon to rest.

5. A sphere of mass 50kg moving at 3m/s overtakes and collides with another sphere of mass 25 kg moving at 1.5m/s in the same direction. Find the velocity of the two masses after impact and loss of kinetic energy during impact in the following cases.

(a) When the impact is inelastic. (b)When the impact is elastic. (c) When the coefficient of restitution is 0.6.

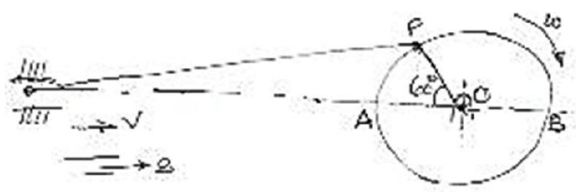
6. The crank pin of an engine has a mass of 1kg and its centre is 150mm from the crank shaft centre. When the shaft is rotating uniformly at 2000 rev/min, calculate in magnitude and direction in each case.

(a) the acceleration of the crank pin

(b) the centripetal force on the crank pin

(c) the centrifugal force on the crank shaft bearings

7. A piston is driven by a crank and connecting rod as shown below. The crank is 75mm long and the rod is 450mm..Assuming the acceleration of the piston to be simple harmonic, find its velocity and acceleration in the position shown when the crank speed is 360 rev/min clockwise. What is the maximum acceleration of the piston and where does it occur?



8. A spring of stiffness 2kN/m is suspended vertically and two equal masses of 4kg each are attached to the lower end. One of these is suddenly removed and the system oscillates. Determine

(a) The amplitude and frequency of the vibration.

(b) The velocity and acceleration of the mass when passing through the half amplitude position, and

(c) The energy of the vibration is joules.