GEC 241- APPLIED MECHANICS II- DYNAMICS
WEEK ONE

KINEMATICS OF PARTICLES

INTRODUCTION TO MECHANICS

Mechanics is a branch of physical science that is concerned with the state of rest of motion of bodies subjected to the action of forces. Engineering mechanics is divided into two areas of study which are statics and dynamics.

From the first aspect of this study, we established that statics is concerned with the equilibrium of a body that is either at rest or moves at a constant velocity. Here we will consider dynamics, which is the study that deals with the accelerated motion of a body.

The subject of dynamics will be presented in two parts:

i. Kinematics-which treats only the geometric aspects of the motion.
ii. Kinetics- which is the analysis of the forces causing the motion.

Dynamics is considered to be more involved than statics since both the forces applied to a body and its motion must be taken into account. There are many problems in engineering whose solutions require application of the principles of dynamics. Typically, the structural design of any vehicle such as automobile or airplane, requires consideration of motion to which it is subjected. This is also true for many mechanical devices, such as motors, pumps, movable tools, industrial manipulators and machinery. Moreover, predictions of the motions of artificial satellites, projectiles, and spacecraft are based on the theory of dynamics.

RECTILINEAR KINEMATICS: CONTINUOUS MOTION

Consider the kinematics of a particle that moves along a rectilinear or straight line path. Recall that a particle has a mass but negligible size and shape. Therefore we must limit application to those objects that have dimensions that are of no consequence in the analysis of the motion. In most problems, we will be interested in bodies of finite size such as rockets, projectiles or vehicles. Each of these objects can be considered as a particle, as long as the motion is characterized by the motion of its mass centre and any rotation of the body is neglected.

Rectilinear kinematics: The kinematics of a particle is characterized by specifying at any given instant, the particle's position, velocity and acceleration.

Position: The origin O on the path is a fixed point and from this point the position coordinate s is used to specify the location of the particle at any given instant. The magnitude of s is the distance from O to the particle usually measured in meters (m) or feet (ft) and the sense of direction is defined by the algebraic sign on s. position is a vector quantity since it has both magnitude and direction.

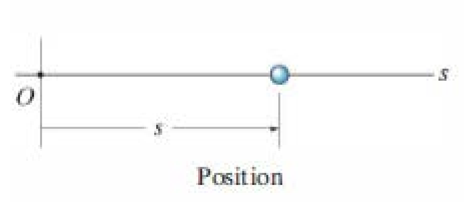


Fig. 1: Position

Displacement: The displacement of the particle is defined as the change in the position. It can be either positive or negative depending on the initial position and the final position. The displacement of a particle is also a vector quantity which is different from distance. Specifically, distance travelled is a positive scalar that represents the total length of path over which the particle travels.

$$\Delta s = s' - s$$

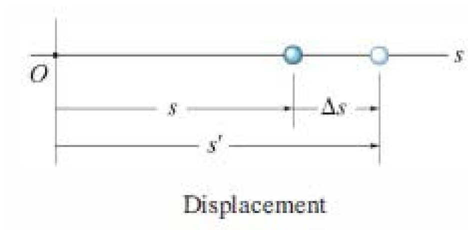


Fig. 2: Displacement

Velocity: It is the rate of change of displacement with time taken. If the particle moves through a displacement Δs during the time interval Δt, the average velocity of the particle during this time interval is

$$v_{ag} = \frac{\Delta s}{\Delta t}$$

The instantaneous velocity is a vector defined as $v = \lim_{\Delta t \to 0}(\Delta s/\Delta t)$, or

$$\boldsymbol{v} = \frac{\boldsymbol{ds}}{\boldsymbol{dt}} \qquad \textbf{(1)}$$

Its unit is m/s or ft/s.

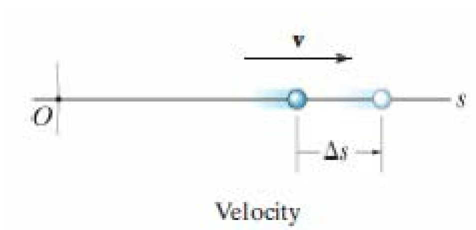


Fig. 3: Velocity

Acceleration: The average acceleration of the particle during the time interval is defined as

$$a_{ag} = \frac{\Delta v}{\Delta t}$$

Where Δv represents the difference in the velocity during the time interval Δt, i.e. $\Delta v = v' - v$

The instantaneous acceleration at time t is a vector that is found by taking smaller and smaller values of Δt and corresponding smaller and smaller values of Δv so that $a = \lim_{\Delta t \to 0}(\Delta v/\Delta t)$ or

$$\boldsymbol{a} = \frac{\boldsymbol{dv}}{\boldsymbol{dt}} \qquad \textbf{(2)}$$

In terms of displacement and time, the instantaneous acceleration is

$$a = \frac{d^2 s}{dt^2}$$

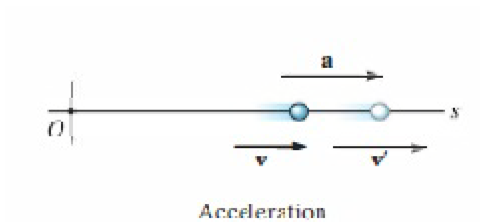


Fig. 4: Acceleration

Both the average and instantaneous acceleration can be either positive or negative. When the particle is slowing down, or its speed is decreasing, the particle is said to be decelerating. The unit is m/s^2 or ft/s^2.

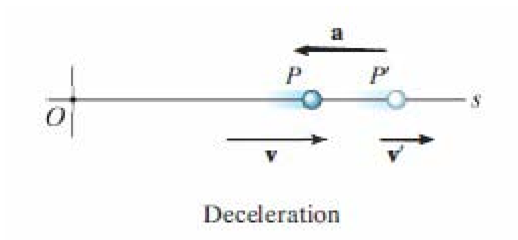


Fig. 5: Deceleration

An important differential relation involving the displacement, velocity and acceleration along the path may be obtained by eliminating the time differential dt in eqns. (1) and (2), we have

$$\boldsymbol{ads = vdv} \qquad \textbf{(3)}$$

Constant Acceleration, $\boldsymbol{a = a_c}$

When the acceleration is constant, each of the three kinematic equations can be integrated to obtain formulas that relate $a_c, v, s\ and\ t$.

Velocity as a Function of Time

Integrate $a_c = {dv}/{dt}$, assuming that initially , $v = v_0$ when $t = 0$.

$$\int_{v_0}^{v} dv = \int_{0}^{t} a_c dt$$

$$\boldsymbol{v = v_0 + a_c t} \qquad \textbf{(4)}$$

Position as a Function of Time

Integrate $v = {ds}/{dt} = v_0 + a_c t$, assuming that initially $s = s_0$ when $t = 0$

$$\int_{s_0}^{s} ds = \int_{0}^{t} (v_0 + a_c t) dt$$

$$\boldsymbol{s = s_0 + v_0 t + \frac{1}{2} a_c t^2} \qquad \textbf{(5)}$$

Velocity as a Function of Position

It is either we solve for t in eqn 4 and substitute into eqn 5 or integrate $vdv = a_c ds$, assuming that $v = v_0 \; at \; s = s_0$.

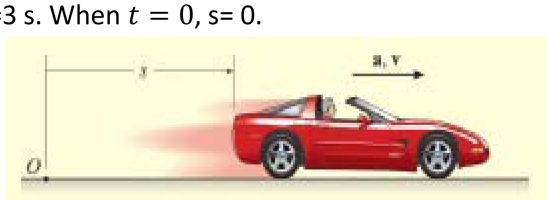
$$\int_{v_0}^{v} vdv = \int_{s_0}^{s} a_c ds$$

$$\boldsymbol{v^2 = {v_0}^2 + 2a_c(s - s_0)} \qquad \textbf{(6)}$$

A typical example of constant accelerated motion occurs when a body falls freely towards the earth. If air resistance is neglected and the distance of fall is short, then the downward acceleration of the body when it is close to the earth is constant and approximately $9.81 m/s^2$.

WORKED EXAMPLES:

1. The car In The figure below moves in a straight line such that for a short time its velocity is defined by $v = (3t^2 + 2t) m/s$, where t is in seconds. Determine its position and acceleration when t=3 s. When $t = 0$, s= 0.



Solution

The position coordinate extends from the fixed origin O to the car, positive to the right.

The car's position can be determined from $\boldsymbol{v} = \frac{\boldsymbol{ds}}{\boldsymbol{dt}}$, since this equation relates $v, s \; and \; t$. noting that s= 0 when t= 0

$$\boldsymbol{v} = \frac{\boldsymbol{ds}}{\boldsymbol{dt}} = (\boldsymbol{3t^2} + \boldsymbol{2t})$$

$$\int_{s_0}^{s} ds = \int_{0}^{t} (3t^2 + 2t)dt$$

$$s = t^3 + t^2$$

When t= 3 s,

$$s = 3^3 + 3^2 = 36 \; m$$

The acceleration is determined from $\boldsymbol{a} = \frac{\boldsymbol{dv}}{\boldsymbol{dt}}$

$$a = \frac{dv}{dt} = \frac{d}{dt}(3t^2 + 2t)$$

= 6t + 2

When t = 3 s,

A = 6(3) +2 = 20 m/s^2

2. A small projectile is fired vertically downward into a fluid medium with an initial velocity of 60m/s. Due to the drag resistance of the fluid the projectile experiences a deceleration of $a = (-0.4v^3)m/s^2$, where v is in m/s. determine the projectile's velocity and position 4 s after it is fired.

SOLUTION

Since the motion is downward, the position coordinate is positive downward

To calculate the velocity, we will apply

$$a = \frac{dv}{dt} = -0.4v^3$$

Integrating with $v_0 = 60m/s$ when $t = 0$

$$\int_{60m/s}^{v} \frac{dv}{-0.4v^3} = \int_{0}^{t} dt$$

$$\frac{1}{-0.4}\left(\frac{1}{-2}\right)\frac{1}{v^2}\bigg|_{60}^{v} = t - 0$$

$$t = \frac{1}{0.8}\left[\frac{1}{v^2} - \frac{1}{60^2}\right]$$

$$v = \left[\frac{1}{60^2} + 0.8t\right]^{-1/2} \; m/s$$

When t = 4 s, $v = 0.559 m/s$

For the position, we will use

$$v = \frac{ds}{dt} = \left[\frac{1}{60^2} + 0.8t\right]^{-1/2}$$

$$\int_0^s ds = \int_0^t \left[\frac{1}{60^2} + 0.8t\right]^{-1/2} dt$$

$$s = \frac{2}{0.8}\left[\frac{1}{60^2} + 0.8t\right]^{1/2}\Bigg|_0^t$$

$$s = \frac{1}{0.4}\left\{\left[\frac{1}{60^2} + 0.8t\right]^{1/2} - \frac{1}{60}\right\} m$$

When t = 4 s, s = 4.43 m

3. During a test a rocket travels upwards at 75 m/s, and when it is 40 m from the ground its engine fails. Determine the maximum height s_B reached by the rocket and its speed just before it hits the ground. While in motion, the rocket is subjected to a constant downward acceleration of 9.81 m/s^2 due to gravity. Neglect the effect of air resistance.

SOLUTION

For the maximum height, since the rocket is travelling upward, $v_A = +75\, m/s$ when t= 0. At the maximum height s = s_B the velocity $v_B = 0$. The acceleration $a_c = -9.81 m/s^2$

$$v_B{}^2 = v_A{}^2 + 2a_c(s_B - s_A)$$
$$0 = (75)^2 + 2(-9.81)(s_B - 40m)$$

$$s_B = 327 \text{ m}$$

To obtain the velocity of the rocket just before it hits the ground

$$v_C{}^2 = v_B{}^2 + 2a_c(s_C - s_B)$$
$$= 0 + 2(-9.81)(0 - 327)$$
$$v_C = -80.1\, m/s$$

ANGULAR DISPLACEMENT

It may be defined as the angle described by a particle from one point to another, with respect to the time. Angular displacement is a vector quantity since it has both magnitude and direction.

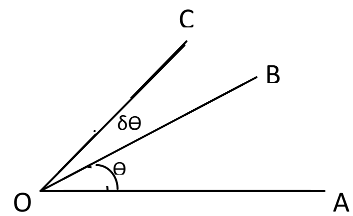


Fig. 5: Angular displacement

For example, let a line OB has its inclination Θradians to the fixed line OA, as shown in the diagram above. If this line moves from OB to OC, through an angle δΘ during a short interval of time δt, then δΘ is known as the angular displacement of the line OB.

ANGULAR VELOCITY

It may be defined as the rate of change of angular displacement with respect to time. It is usually expressed mathematically as

$$\omega = \frac{d\theta}{dt} \qquad (7)$$

Since it has magnitude and direction, therefore it is a vector quantity. Note that if the direction of the angular displacement is constant, then the rate of change of magnitude of the angular displacement with respect with time is termed angular speed.

ANGULAR ACCELERATION

It may be defined as the rate of change of angular velocity with respect to time. Mathematically, angular acceleration is given as

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2} \qquad (8)$$

It is also a vector quantity.

EQUATIONS OF ANGULAR MOTION

It is important to note that the equations of angular motion are similar to that of the linear motion.

i. $\omega = \omega_0 + \alpha.t$ (9)

ii. $\theta = \omega_0 + \frac{1}{2}\alpha.t^2$ (10)

iii. $\omega^2 = (\omega_0)^2 + 2\alpha.\theta$ (11)

iv. $\theta = \frac{(\omega_0+\omega)t}{2}$ (12)

Where ω_0= Initial angular velocity in rad/s

ω= Final angular velocity in rad/s

t = Time in seconds

θ =Angular displacement in time t seconds

α = Angular acceleration in rad/s^2

Note: If a body is rotating at the rate of N r.p.m. (revolution per minute), then its angular velocity,

$$\omega = \frac{2\pi N}{60} \text{ rad/s} \qquad (14)$$

RELATION BETWEEN LINEAR AND ANGULAR QUANTITIES OF MOTION

Consider a body moving along a circular path from A to B

Let r = radius of the circular path

- θ= angular displacement in radians
- s = linear displacement
- v = linear velocity
- ω = angular velocity
- a =linear acceleration
- α = angular acceleration

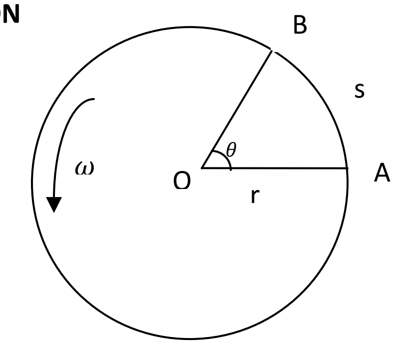


Fig. 6: Motion of a body along a circular path

From the geometry, we know that

$$s = r.\theta \qquad (15)$$

We also know that the linear velocity,

$$v = \frac{ds}{dt} = \frac{d(r.\theta)}{dt} = r \times \frac{d\theta}{dt} = r.\omega \qquad (16)$$

And linear acceleration,

$$a = \frac{dv}{dt} = \frac{d(r.\omega)}{dt} = r \times \frac{d\omega}{dt} = r.\alpha \qquad (17)$$

Example

1. A wheel accelerates uniformly from rest to 200 r.p.m in 20 seconds. What is its angular acceleration? How many revolutions does the wheel make in attaining the speed of 2000 r.p.m.?

   Solution

   Given: N_0=0 0r ω = 0, N = 2000 r.p.m or $\omega = \frac{2\pi \times 2000}{60} = 209.5$ rad/s, t = 20 s

   Angular acceleration can be calculated using the formula

$$\omega = \omega_0 + \alpha.t$$

$$209.5 = 0 + \alpha \times 20$$

$$\alpha = \frac{209.5}{20} = 10.475 \text{rad/s}^2$$

Number of revolution made by the wheel

$$\theta = \frac{(\omega_0 + \omega)t}{2} = \frac{(0 + 209.5)20}{2} = 2095\ rad$$

Since the angular distance moved by the wheel during one revolution is 2π radians, therefore the number of revolutions made by the wheel,

$$n = \frac{\theta}{2\pi} = \frac{2095}{2\pi} = 333.4$$

PRACTICE QUESTIONS

1. Initially, a car travels along a straight road with a speed of 35 m/s. if the brakes are applied and the speed of the car is reduced to 10 m/s in 15 s, determine the constant deceleration of the car.
2. The position of a particle is given by $s = (2t^2 - 8t + 6)m$, where t is in seconds. Determine the time when the velocity of the particle is zero, and the total distance travelled by the particle when t = 3 s.
3. A particle travels along a straight line with a velocity of $v = (20 - 0.05s^2)$ m/s, where s is in meters. Determine the acceleration of the particle at s = 15 m.
4. A train starts from rest at a station and travels with a constant acceleration of 1m/s^2. Determine the velocity of the train when t = 30s and the distance travelled during this time.
5. The position of a particle along a straight line is given by $s = (1.5t^3 - 13.5t^2 + 22.5t)ft$, where t is in seconds. Determine the position of the particle when t = 6 s and the total distance it travels during the 6s time interval.
6. The angular displacement of a body is a function of time and is given by equation:

   $$\theta = 10 + 3t + 6t^2, where\ t\ is\ in\ seconds$$

   Determine the angular velocity, displacement and acceleration when t = 5 seconds.