TYPES OF SIGNAL TRANSMISSION

The electrical signals produced by encoders are of two types, namely analogue signals and digital signals. These two types of signals results in two types of signal transmission:

1. **Analogue signal transmission** which is used in a communication that involves the transmission of analogue signals from the transmitter to the receiver. Analogue signals continuously vary with time. They are sinusoidal in nature and usually have harmonics. They represent the variations of physical quantities such as sound, pressure, temperature, etc. and are represented by voltage waveforms that have different amplitudes at different instants of time. Examples of analogue signal transmissions are voice transmission through a telephone line, Radio and TV broadcast to the general public. Sometimes analogue signals are first converted into digital signals before being transmitted.

2. **Digital signal transmission** which is used in a communication that involves the transmission of digital signals from the transmitter to the receiver. Digital signals are not continuous. They are made up of pulses which occur at discrete intervals of time. The pulses may occur singly at a definite period of time or as a coded group. These signals play a very important role in the transmission and reception of coded messages. Examples of digital signals are

    a. **Telegraph signal** which is generated by a telegraph and teleprinter which are the most common instruments being used to transmit written text in the form of coded signals.

    b. **Radar signal** which is generated by a radar (a device being used to find out the location of distant objects in terms of location and bearing by transmitting a short period signal and beaming it to the location of the target. The reflected signal is picked up by the radar

    c. **Data signals** which are generated by several devices and are required to transmit data from one place to another. The data to be transmitted are converted into electrical pulses before transmission is done.

SIGNAL SPECTRUM

Plotting the amplitude of a signal at various instants of time is used to represent the signal in the ***Time domain*.** Plotting the amplitudes of the different frequency components is termed the ***frequency domain*** representation. This plot gives the spectral component amplitudes of the signal against frequency.

Analogue signals when analysed are found to comprise of certain fundamental frequencies and their harmonics. They occupy only a small portion of the frequency spectrum which is termed as the ***Discrete spectrum.***

The analysis of digital signals on the other hand gives an infinite number of frequencies. Such a spectrum is termed as ***Continuous spectrum.***

SIGNAL ANALYSIS

Signals are single valued functions of time (t) and are of complex nature. No matter how complex a signal wave form may be, it comprises of one or more sine and / or cosine functions. Assume that we have a square wave given by the expression 1.1

$$f(x) = \begin{Bmatrix} 1(0 < t < \pi) \\ -1(\pi > t < 2\pi) \end{Bmatrix} \ldots\ldots 1.1$$

The signal is represented by the figure 1.

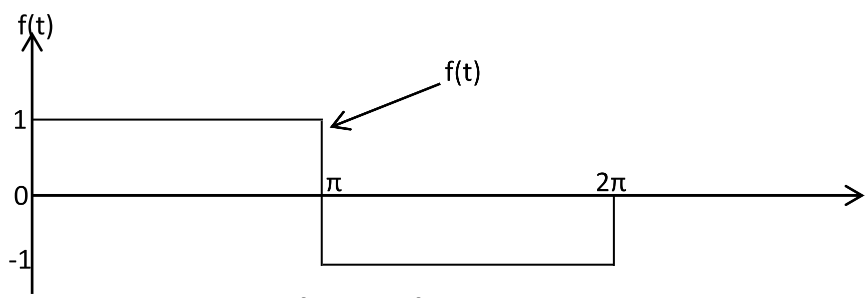

Fig. 1a A square wave function f(t).

Let us try to see how a sine function of the same time period can be used to represent this square wave form.

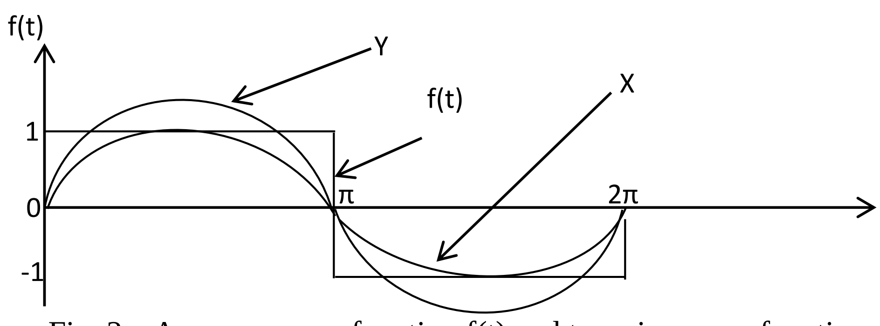

Fig. 2a. A square wave function f(t) and two sine wave functions.

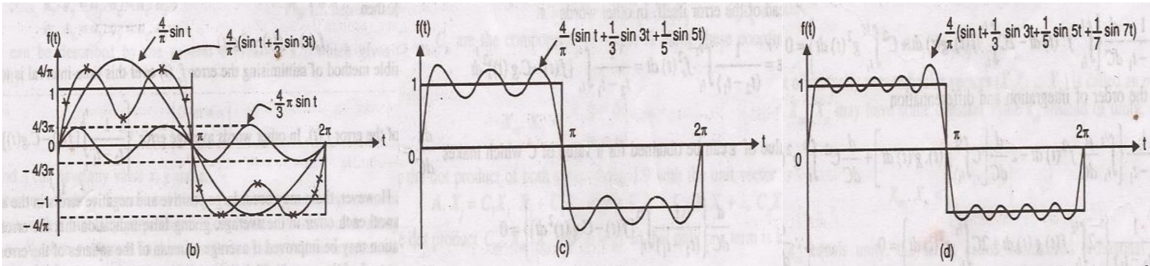

Fig2(b), (c), (d): A square wave function approximated by a sine wave function.

In Fig2a, we introduced a sine function marked X, having the same peak magnitude as the square wave f(t), hence its magnitude is equal to the square wave only at the peak point indicating that it is a very poor representation or approximation of the square wave.

If the magnitude of the sine wave X, is increased as shown by the sine wave Y, its magnitude becomes equal to the sine wave magnitude at two points. This provides an approximation slightly better than the first curve even though it is still a very poor approximation.

In figure 2b, another sine wave component is added to improve the approximation. This component has a frequency thrice the first component. It is easily seen that this provides a better approximation. The approximated wave approaches more closely to the square wave when more sine wave components are added. As shown in Figure 2c-2d.

The graphical method of approximating one function to another gives a clear understanding but is difficult to use in practice, hence it is always necessary to use analytical methods of approximating the square wave function with a sine wave function.

Consider two signals f(t) and g(t). Assume that f(t) is to be approximated in terms of g(t) over the interval (t_1-t_2). This approximation may be written as;

$f(t) \cong C.g(t)$ for ($t_1<t<t_2$)……………………………………………………..2

Where C is a constant and has a value such that error between the actual function and the approximated function is minimum over the time-interval considered. If the error function if denoted as $f_e(t) = f(t) - C.g(t)$ … .3

One possible method of minimizing the error $f_e(t)$ over this time-interval is to minimise the average value of the error $f_e(t)$. In other words average error $\left\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2}[f(t) - C.g(t)]dt\right\}$ should be kept minimum. However, there may occur large positive and negative errors in the approximation which cancel each other in the average giving false indication that the error is minimum.

The situation may be improved if average or mean of the squares of the error denoted by ε is minimised, instead of the error itself. In other words,

$$\varepsilon = \left\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2} f_e^2(t)dt\right\} = \left\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2}[f(t) - C.g(t)]^2 dt\right\} \qquad \ldots 4$$

Minimum value of ε can be obtained for a value of C which makes $\frac{d\varepsilon}{dC} = 0$

Or $\frac{d}{dC}\left\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2}[f(t) - C.g(t)]^2 dt\right\} = 0$

Therefore, $\frac{1}{t_2-t_1}\frac{d}{dC}\left\{\int_{t_1}^{t_2} f^2(t)dt - 2C\int_{t_1}^{t_2} f(t).g(t)dt + C^2\int_{t_1}^{t_2} g^2(t)\,dt\right\} = 0$

Interchanging the order of integration and differentiation

$$\frac{1}{t_2-t_1}\left\{\int_{t_1}^{t_2}\frac{d}{dC}f^2(t)dt - 2\frac{d}{dC}\left[C\int_{t_1}^{t_2} f(t).g(t)dt\right] + \frac{d}{dC}C^2\int_{t_1}^{t_2} g^2(t)\,dt\right\} = 0$$

But $\frac{d}{dC}f^2(t)dt = 0$,

Therefore the expression becomes,

$$\frac{1}{t_2-t_1}\left\{-2\int_{t_1}^{t_2} f(t).g(t)dt + 2C\int_{t_1}^{t_2} g^2(t)\,dt\right\} = 0$$

Therefore $C = \frac{\int_{t_1}^{t_2} f(t).g(t)dt}{\int_{t_1}^{t_2} g^2(t)dt}$ …5

Equation 5 gives the value of C for obtaining the best approximation.

ORTHOGONAL FUNCTIONS

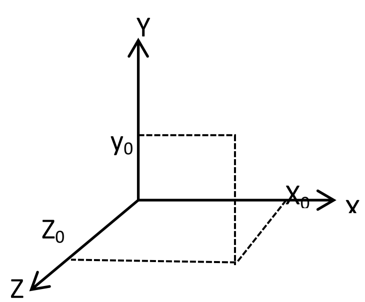

Fig. 3. Representation of a vector in 3 coordinates

The concept of orthogonality can be understood by considering the example of Vector **A** represented by Fig 3.

If $\mathbf{a}_x$, $\mathbf{a}_y$ and $\mathbf{a}_z$ are the unit vectors along x, y and z axes, then vector components along the three axes are $x_0. \mathbf{a}_x$, $y_0. \mathbf{a}_y$ and $z_0. \mathbf{a}_z$ respectively so that

$A= x_0. \mathbf{a}_x + y_0. \mathbf{a}_y + z_0. \mathbf{a}_z$……………………………………………………………6

Since the three vector are mutually perpendicular, the dot product will be

$$a_x. a_y = a_y. a_z = a_z. a_x = 0$$

and

$$a_x. a_x = a_y. a_y = a_z. a_z = 1$$

This can be described by the general expression of equation 7 which gives the condition for orthogonality.

$$a_m. a_n = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$ ……………………………………………………………………7

where *m* and *n* can have any value *x*, *y* and *z*.

The concept of three dimensional vector representation may be extended to *n*-dimensional representation with *n*-mutually perpendicular coordinates. If unit vectors along these coordinates are X_1, X_2 ......X_n, then a vector *A* in this coordinate system can be represented as:

$$A = C_1X_1 + C_2X_2 + C_3X_3 + ... C_nX_n +$$ ……………………………………………………8

where C_1, C_2 .... C_n are the components of vector *A* along these coordinates.

Equation 6 may be rewritten as

$$X_m. X_n = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$ ………………………………………………………………………9

Taking the dot product of both sides of Eq. 1.9 with the unit vector X_j, we obtain

$$A. X_j = C_1X_1. X_j + C_2X_2. C_3X_3. X_j + ... C_j. X_j X_j + ... C_nX_nX_j$$

But the dot product $C_m. X_m. X_j = 0\ (m \neq j)$ so that only one term is left on the right side of this expression

$$AX_j = C_jX_jX_j = C_j$$ ……………………………………………………………………10

The set of mutually perpendicular vectors $(X_1X_2 ... X_n)$ is called an *orthogonal vector space*.

The product $\mathbf{X}_m.\mathbf{X}_n$ may have some constant value $\mathbf{k}_m$ instead of unity so that Eq. 9 may be rewritten as

$$X_m. X_n = \begin{cases} 0 & m \neq n \\ k_m & m = n \end{cases}$$ ……………………………………………………………………11.

Where $\mathbf{k}_m$ equals unity, the set is called normalized orthogonal set. Eq. 10 may be modified as

$$A. X_j = C_jX_j. X_j = C_jK_j$$

or

$$C_j = A. \frac{X_j}{K_j}$$

To summarise, we may say that orthogonal vector space comprises mutually perpendicular vector components. If a vector is orthogonal to another vector, it has no component along the other. Similarly, if a function is orthogonal to another function, it does not contain any component or the form of the other function. *A function cannot be approximated to another function orthogonal to it.* If we try to approximate a function with its orthogonal function, the error will be larger than the original function. This is similar to orthogonal vector components a_x and a_y. There can be no component of a_x in a_y and vice-versa.

When two signal are orthogonal

$$C_j = 0$$

Equation 1.5 then becomes

$$C = \frac{\int_{t_1}^{t_2} f(t)\, g(t)\, dt}{\int_{t_1}^{t_2} g^2(t) f\, dt} = 0$$

or

$$\int_{t_1}^{t_2} f(t)\, g(t) = 0 \qquad \text{... (1.13)}$$

Example 1.1. *Show that the functions sin nwt and cos mwt are orthogonal over the interval* $\left(t_0, t_0 + \frac{2\pi}{\omega_0}\right)$ *where n and m are any integers.*

Solution. Let $f(t) = \sin n\ \omega_0\ t$

and $g(t) = \cos m\ \omega_0\ t$

$$C = \int_{t_1}^{t_2} f(t).\ g(t)\ dt$$

$$= \int_{t_0}^{t_0 + \frac{2\pi}{\omega_0}} [\sin n\ \omega_0\ t \cos m\ \omega_0 t.\ dt]$$

$$= \int_{t_0}^{t_0 + \frac{2\pi}{\omega_0}} \frac{1}{2}(2 \sin n\ \omega_0\ t \cos m\ \omega_0 t)\ dt$$

$$= \frac{1}{2} \int_{t_0}^{t_0 + \frac{2\pi}{\omega_0}} [\sin (n-m)\ \omega_0 t + \sin (n+m)\ \omega_0 t]\ dt$$

$$= -\frac{1}{2\omega_0} \left[\frac{1}{(n-m)} \cos (n-m)\ \omega_0 t + \frac{1}{(n+m)} \cos (n+m)\ \omega_0 t \right]_{t_0}^{t_0 + \frac{2\pi}{\omega_0}}$$

$$= 0$$

Since $C = 0$, the two functions are orthogonal.

Similarly, it can be shown that $\sin n\ \omega t$ and $\sin m\ \omega t$ and also $\cos n\ \omega t$ and $\cos m\ \omega t$ are orthogonal.

Example 1.2. *Determine the magnitude of the curve marked Y in Fig. 1.6 to ensure that mean square error is minimum when approximated to the square wave function.*

Solution. Let the square and sine wave functions be represented as

$$f(t) = \begin{cases} 1 & (0 < t < \pi) \\ -1 & (\pi < t < 2\pi) \end{cases}$$

$g(t) = \sin t$ respectively

$f(t)$ will be approximated with $g(t)$ over the period (0 to 2π)

$$\therefore \quad f(t) \cong C \sin t$$

Optimum value of C to minimise the mean square error in the approximation is given as

$$C = \frac{\int_0^{2\pi} f(t) \sin t\ dt}{\int_0^{2\pi} \sin^2 t\ dt}$$

$$= \frac{\int_0^{\pi} \sin t\ dt - \int_{\pi}^{2\pi} \sin t\ dt}{\int_0^{2\pi} \left(\frac{1 - \cos 2t}{2} \right) dt}$$

$$= \frac{-[\cos t]_0^{\pi} + [\cos t]_{\pi}^{2\pi}}{\left[1 - \frac{1}{2} \sin 2t \right]_0^{2\pi}}$$

$$= \frac{4}{\pi} \quad \textbf{Ans.}$$

ORTHOGONALITY IN COMPLEX FUNCTIONS

The previous discussion of orthogonality was limited to functions of real variables. If $f(t)$ and $g(t)$ are complex functions of real variable t, then $f(t)$ may be approximated over the interval (t_1, t_2) as

$$f(t) \cong Cg(t)$$

It can be shown that the optimum value of C to ensure minimum mean square error is given as

$$C = \frac{\int_{t_1}^{t_2} f(t)\, g^*(t)\, dt}{\int_{t_1}^{t_2} g(t)\, g^*(t)\, dt} \quad \text{... (1.14)}$$

where $g^*(t)$ is conjugate of $g(t)$.

The condition for orthogonality of the two functions is obtained by equating Eq. 1.14 to zero.

i.e.,

$$\int_{t_1}^{t_2} f(t)\, g^*(t)\, dt = \int_{t_1}^{t_2} f^*(t)\, g(t) = 0$$

where $f^*(t)$ is conjugate of $f(t)$.

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For a mutually orthogonal set of complex functions over the interval (t_1, t_2), condition for orthogonality is given by Eq. 1.15

$$\int_{t_1}^{t_2} f_m(t) f_n^*(t)\, dt = \begin{Bmatrix} 0 & m \neq n \\ 1 & m = n \end{Bmatrix} \quad \text{... [1.15 (a)]}$$

If $g(t)$ is a complete set of function, then function $f(t)$ can be expressed as

$$f(t) = C_1 g_1(t) + C_2 g_2(t) + \ldots + C_n g_n(t) \quad \text{... [1.15 (b)]}$$

where

$$C_n = \frac{1}{K_n} \int_{t_1}^{t_2} f(t)\, g_n^*(t)\, dt$$

It should be remembered that in case of real functions $g^*(t) = g(t)$.

APPROXIMATING A FUNCTION BY A SET OF MUTUALLY ORTHOGONAL FUNCTIONS

Assume a set of function,

$g_1(t), g_2(t) \ldots g_n(t)$ orthogonal to one another over an interval of t_1 to t_2.

We know that

$$g_j(t)\, g_k(t) = \begin{bmatrix} 0 & j \neq k \\ K_j & j = k \end{bmatrix}$$

When

$$j = k = n,\ g_n^2(t) = K_n$$

If a function $f(t)$ is to be approximated over an interval (t_1 to t_2) by the above set of functions, then

$$f(t) \cong C_1 g_1(t) + C_2 g_2(t) + \ldots C_n g_n(t)$$

$$= \sum_1^n C_n\, g_n(t) \quad \text{... (1.16)}$$

The mean square error ε is given as

$$\varepsilon = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} [f(t) - \sum_1^n C_n g_n(t)]^2 dt \quad \text{... (1.17)}$$

Since, the mean square error ε is function of constants $C_1, C_2 \ldots C_n$, minimum value of e will be obtained when

$$\frac{\delta\varepsilon}{\delta C_1} = \frac{\delta\varepsilon}{\delta C_2} = \ldots\ldots\ldots \frac{\delta\varepsilon}{\delta C_n} = 0$$

It can be easily shown that minimum value of mean square error will be obtained when

$$C_j = \frac{\int_{t_1}^{t_2} f(t) g_j(t)\, dt}{\int_{t_1}^{t_2} g_j^2(t)\, dt} \quad \text{... (1.18)}$$

$$= \frac{1}{K_j}\int_{t_1}^{t_2} f(t)\, g_j(t)\, dt \qquad \text{... (1.19)}$$

In order to obtain the best approximation with minimum mean square error, the coefficients $C_1, C_2 \ldots C_n$ should be chosen as given in Eq. 1.19.

EVALUATION OF MEAN SQUARE ERROR

The mean square error “e” given is determined by the use of equation 1.17.

$$\varepsilon = \frac{1}{(t_2 - t_1)}\int_{t_1}^{t_2}\left[f(t) - \sum_1^n C_n g_n(t)\right]^2 dt$$

$$\varepsilon = \frac{1}{(t_2 - t_1)}\left[\int_{t_1}^{t_2} f^2(t)\, dt + \sum_1^n \int_{t_1}^{t_2} C_n^2 g_n^2(t)\, dt - 2\sum_1^n \int_{t_1}^{t_2} C_n f(t) g_n(t)\, dt\right] \qquad \text{... (1.20)}$$

But from Eq. 1.19: $\int_{t_1}^{t_2} f(t) g_n(t)\, dt = C_n \cdot K_n$

Also $\int_{t_1}^{t_2} g_n^2(t) dt = K_n$. Substituting these values in Eq. 1.20.

$$\varepsilon = \frac{1}{(t_2 - t_1)}\left[\int_{t_1}^{t_2} f^2(t)\, dt + \sum_1^n C_n^2 K_n - 2\sum_1^n C_n^2 K_n\right]$$

$$= \frac{1}{(t_2 - t_1)}\left[\int_{t_1}^{t_2} f^2(t)\, dt - \sum_1^n C_n^2 K_n\right] \qquad \text{... (1.21)}$$

$$= \frac{1}{(t_2 - t_1)}\left[\int_{t_1}^{t_2} f^2(t)\, dt - (C_1^2 K_1 + c_2^2 K_2 + \ldots C_n^2 K_n)\right] \qquad \text{... (1.22)}$$

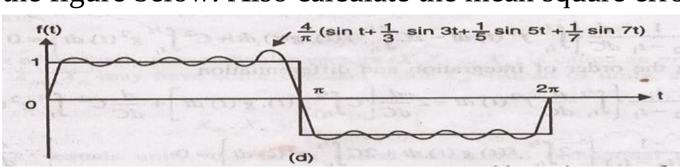
Equation. 1.22 shows that the mean square error can be decreased by increasing n in the approximation. When n is increased to infinity, the error becomes zero. Under this condition

$$\int_{t_1}^{t_2} f^2(t)\, dt = \sum_{n=1}^{n=\infty} C_n^2 \cdot K_n \qquad \text{... (1.23)}$$

and $\qquad f(t) = C_1 g_1(t) + C_2 g_2(t) \ldots + C_n g_n(t)$: $(n \to \infty)$ ... (1.24)

The series is said to converge in the mean. Eq. 1.24 is known as the Generalised *Fourier Series representation* of the function $f(t)$.

Assignment: Determine the values of constants C1, C2, …….C7 in the approximated waveform of the figure below. Also calculate the mean square error.



Solution. The approximated function may be written as

$$f(t) = C_1 \sin t + C_2 \sin 2t + C_3 \sin 3t + \ldots\, C_7 \sin 7t$$

The value of constant C_n is determined by the use of Eq. 1.18.

(*a*)
$$C_n = \frac{\int_0^{2\pi} f(t) \sin nt\, dt}{\int_0^{2\pi} \sin^2 nt\, dt}$$

$$= \frac{\left(\int_0^{\pi} \sin nt\, dt - \int_{\pi}^{2\pi} \sin nt\, dt\right)}{\int_0^{2\pi} \frac{1 - \cos nt}{2}\, dt}$$

$$= \frac{-\frac{1}{n}[\cos nt]_0^{\pi} - \frac{1}{n}[\cos nt]_{\pi}^{2\pi}}{\frac{1}{2}\left[t - \frac{\sin nt}{n}\right]_0^{2\pi}}$$

$$= \frac{-\frac{1}{n}[\cos nt]_0^{\pi} - \frac{1}{n}[\cos nt]_{\pi}^{2\pi}}{\frac{1}{2}\left[t - \frac{\sin nt}{n}\right]_0^{2\pi}}$$

$$f(t) = \frac{4}{\pi}\sin t + \frac{4}{3\pi}\sin t + \frac{4}{5\pi}\sin 5t + \frac{4}{7\pi}\sin 7t$$

$$= \frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \frac{1}{7}\sin 7t\right)$$

(*b*) The mean square error ε is given by Eq. 1.17

$$\varepsilon = \frac{1}{(t_2 - t_1)}\left[\int_{t_1}^{t_2} f^2(t)\,dt - \sum_1^n C_n^2 K_n\right] dt$$

Now (*i*) $t_2 - t_1 = 2\pi$

$$f(t) = \begin{Bmatrix} 1 & (0 < t < \pi) \\ -1 & (\pi < t < 2\pi) \end{Bmatrix}$$

(*ii*) $$\int_0^{2\pi} f^2(t)\,dt = 2\pi$$

(*iii*) $$C_n = \begin{Bmatrix} \frac{4}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{Bmatrix}$$

(*iv*) $$K_n = \int_0^{2\pi} \sin^2 nt\,dt = \pi$$

$\therefore$ $$\varepsilon_1 = \frac{1}{2\pi}\left[2\pi - \left(\frac{4}{\pi}\right)^2 \times \pi\right] = 0.19$$

$$\varepsilon_3 = \frac{1}{2\pi}\left(2\pi - \left(\frac{4}{\pi}\right)^2 \times \pi - \left(\frac{4}{3\pi}\right)^2 \times \pi = 0.1\right.$$

$$\varepsilon_5 = \frac{1}{2\pi}\left[2\pi - \left(\frac{4}{\pi}\right)^2 \times \pi - \left(\frac{4}{3\pi}\right)^2 \times \pi - \left(\frac{4}{5\pi}\right)^2 \times \pi\right]$$
$$= 0.0675$$

$$\varepsilon_7 = \frac{1}{2\pi}\left[2\pi - \left(\frac{4}{\pi}\right)^2 \times \pi - \left(\frac{4}{3\pi}\right)^2 \times \pi - \left(\frac{4}{5\pi}\right)^2 \times \pi - \left(\frac{4}{7\pi}\right)^2 \times \pi\right]$$
$$= 0.051$$

It can be summarised from the example above that as the number of terms is increased in the approximation, the mean square error is reduced. Representation of a function over an interval by a linear set of mutually orthogonal functions is termed the *Fourier Series representation*. Since there exists a large number of sets of orthogonal functions, a function may be represented in terms of different sets of orthogonal functions. This is analogous to the representation of a vector using different coordinate systems.